« Iterated Forward/Backward Relational Infinitary Static Analysis »

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METHODES ITERATIVES DE CONSTRUCTION
ET D'APPROXIMATION DE PCINTS FIXES
D'OPERATEURS MONOTONES SUR UN TREILLIS,
ANALYSE SEMANTIQUE DES PROGRAMMES.

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Recalls on operational and collecting semantics

Programs states (recall from lecture 5)

States $\langle \ell, \rho \rangle \in \Sigma[\![P]\!]$ record a program point $\ell \in \mathrm{in}_P[\![P]\!]$ and an environment $\rho \in \mathrm{Env}[\![P]\!]$ assigning values to variables:

$$\Sigma \in \mathrm{Prog} \mapsto \wp(\mathbb{V} \times \mathbb{R}),$$
$$\Sigma[\![P]\!] \stackrel{\mathrm{def}}{=} \mathrm{in}_P[\![P]\!] \times \mathrm{Env}[\![P]\!] . \qquad (1)$$

Program transition relation (recall from lecture 5)

The small-step operational semantics of commands, sequences and programs $C \in \mathrm{Com} \cup \mathrm{Seq} \cup \mathrm{Prog}$ within a program $P \in \mathrm{Prog}$ involves transition judgements [1]

$$\langle \ell,\, \rho\rangle \models\!\!\llbracket C \rrbracket\!\!\Longrightarrow \langle \ell',\, \rho'\rangle\ .$$

[1] Such judgements mean that if execution is at control point $\ell \in \mathrm{in}_P\llbracket C\rrbracket$ in environment $\rho \in \mathrm{Env}\llbracket P\rrbracket$ then the next computation step within command C leads to program control point $\ell' \in \mathrm{in}_P\llbracket C\rrbracket$ in the new environment $\rho' \in \mathrm{Env}\llbracket P\rrbracket$.

Transition system of a program (recall from lecture 5)

The transition system of a program $P = S\ ;;$ is

$$\langle \Sigma\llbracket P\rrbracket,\, \tau\llbracket P\rrbracket\rangle$$

where $\Sigma\llbracket P\rrbracket$ is the set (1) of program states and $\tau\llbracket C\rrbracket$, $C \in \mathrm{Cmp}\llbracket P\rrbracket$ is the transition relation for component C of program P, defined by

$$\tau\llbracket C\rrbracket \stackrel{\mathrm{def}}{=} \{\langle\langle \ell,\, \rho\rangle,\, \langle \ell',\, \rho'\rangle\rangle \mid \langle \ell,\, \rho\rangle \models\!\!\llbracket C \rrbracket\!\!\Longrightarrow \langle \ell',\, \rho'\rangle\} \quad (2)$$

Initial States (recall from lecture 5)

Execution starts at the program entry point with all variables uninitialized:

$$\mathrm{Entry}\llbracket P\rrbracket \stackrel{\mathrm{def}}{=} \{\langle \mathrm{at}_P\llbracket P\rrbracket,\, \lambda \mathtt{X} \in \mathrm{Var}\llbracket P\rrbracket \cdot \Omega_{\mathrm{i}}\rangle\}\ . \quad (3)$$

Final States (recall from lecture 5)

Execution ends without error when control reaches the program exit point

$$\mathrm{Exit}\llbracket P\rrbracket \stackrel{\mathrm{def}}{=} \{\mathrm{after}_P\llbracket P\rrbracket\} \times \mathrm{Env}\llbracket P\rrbracket\ .$$

When the evaluation of an arithmetic or boolean expression fails with a runtime error, the program execution is blocked so that no further transition is possible.

Big-step operational semantics of a program (recall from lecture 5)

- The big-step operational semantics of a program P is

$$\langle \Sigma[\![P]\!],\ t^*[\![P]\!] \rangle$$

  where $t^*[\![P]\!] \stackrel{\text{def}}{=} (t[\![P]\!])^*$ is the reflexive transitive closure of the transition relation $t[\![P]\!]$
- Infinite executions are not considered with this semantics

Structural big-step operational semantics (recall from lecture 8)

$$\tau^\star[\![\texttt{skip}]\!] = 1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{skip}]\!] \tag{4}$$

$$\tau^\star[\![\texttt{X := } A]\!] = 1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{X := } A]\!]$$

$$\begin{aligned}\tau^\star[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!] = {} & (1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t) \ \cup \\ & (1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f)\end{aligned} \tag{5}$$

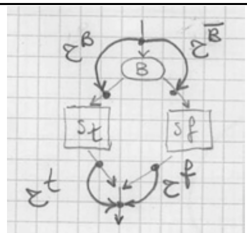
where:

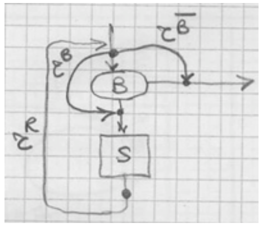
$$\tau^B \stackrel{\text{def}}{=} \{\langle\langle \mathsf{at}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle,\ \langle \mathsf{at}_P[\![S_t]\!],\ \rho\rangle\rangle \mid \rho \vdash B \Rightarrow \mathtt{tt}\}$$

$$\tau^{\bar{B}} \stackrel{\text{def}}{=} \{\langle\langle \mathsf{at}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle,\ \langle \mathsf{at}_P[\![S_f]\!],\ \rho\rangle\rangle \mid \rho \vdash T(\neg B) \Rightarrow \mathtt{tt}\}$$

$$\tau^t \stackrel{\text{def}}{=} \{\langle\langle \mathsf{after}_P[\![S_t]\!],\ \rho\rangle,\ \langle \mathsf{after}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle\rangle \mid \rho \in \mathrm{Env}[\![P]\!]\}$$

$$\tau^f \stackrel{\text{def}}{=} \{\langle\langle \mathsf{after}_P[\![S_f]\!],\ \rho\rangle,\ \langle \mathsf{after}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle\rangle \mid \rho \in \mathrm{Env}[\![P]\!]\}$$





$$\begin{aligned}\tau^\star[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!] = {} & ((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ \\ & (1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}})) \cup \tau[\![S]\!]^\star\end{aligned} \tag{6}$$

where:

$$\tau^B \stackrel{\text{def}}{=} \{\langle\langle \mathsf{at}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle,\ \langle \mathsf{at}_P[\![S]\!],\ \rho\rangle\rangle \mid \rho \vdash B \Rightarrow \mathtt{tt}\}$$

$$\tau^{\bar{B}} \stackrel{\text{def}}{=} \{\langle\langle \mathsf{at}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle,\ \langle \mathsf{after}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle\rangle \mid \rho \vdash T(\neg B) \Rightarrow \mathtt{tt}\}$$

$$\tau^R \stackrel{\text{def}}{=} \{\langle\langle \mathsf{after}_P[\![S]\!],\ \rho\rangle,\ \langle \mathsf{at}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle\rangle \mid \rho \in \mathrm{Env}[\![P]\!]\}$$

Definition of the forward collecting semantics of boolean expressions (recall from lecture 8)

Recall the *collecting semantics* $\mathrm{Cbexp}[\![B]\!]R$ of a boolean expression B from course 8:

$$\mathrm{Cbexp} \in \mathrm{Bexp} \mapsto \wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathrm{Env}[\![P]\!]),$$
$$\mathrm{Cbexp}[\![B]\!]R \stackrel{\mathrm{def}}{=} \{\rho \in R \mid \rho \vdash B \Mapsto \mathtt{tt}\} . \qquad (7)$$

such that:

$$\mathrm{Cbexp}[\![B]\!]\Big(\bigcup_{k\in\mathcal{S}} R_k\Big) = \bigcup_{k\in\mathcal{S}} (\mathrm{Cbexp}[\![B]\!]R_k)$$
$$\mathrm{Cbexp}[\![B]\!]\emptyset = \emptyset .$$

Structural specification of the forward collecting semantics of boolean expressions (recall from lecture 8)

$$\mathrm{Cbexp}[\![\mathtt{true}]\!]R \stackrel{\mathrm{def}}{=} R$$
$$\mathrm{Cbexp}[\![\mathtt{false}]\!]R \stackrel{\mathrm{def}}{=} \emptyset$$
$$\mathrm{Cbexp}[\![A_1 \ \mathsf{c}\ A_2]\!] \stackrel{\mathrm{def}}{=} \underline{\mathsf{c}}^{C}\ (\mathrm{Faexp}[\![A_1]\!], \mathrm{Faexp}[\![A_2]\!])R$$
$$\text{where } \underline{\mathsf{c}}^{C}\ (F,G)R \stackrel{\mathrm{def}}{=} \{\rho \in R \mid \exists v_1 \in F(\{\rho\}) \cap \mathbb{I} : \exists v_2 \in G(\{\rho\}) \cap \mathbb{I} : v_1\ \underline{\mathsf{c}}\ v_2 = \mathtt{tt}\}$$
$$\mathrm{Cbexp}[\![B_1 \ \&\ B_2]\!]R \stackrel{\mathrm{def}}{=} \mathrm{Cbexp}[\![B_1]\!]R \cap \mathrm{Cbexp}[\![B_2]\!]R$$
$$\mathrm{Cbexp}[\![B_1 \mid B_2]\!]R \stackrel{\mathrm{def}}{=} \mathrm{Cbexp}[\![B_1]\!]R \cup \mathrm{Cbexp}[\![B_2]\!]R$$

Backward collecting semantics of arithmetic expressions (recall)

The *backward/top-down collecting semantics* $\mathrm{Baexp}[\![A]\!](R)P$ of an arithmetic expression A defines the subset of possible environments R such that the arithmetic expression may evaluate, without producing a runtime error, to a value belonging to given set P:

$$\mathrm{Baexp} \in \mathrm{Aexp} \mapsto \wp(\mathbb{R}) \stackrel{\sqcup}{\longmapsto} \wp(\mathbb{I}_\Omega) \stackrel{\sqcup}{\longmapsto} \wp(\mathbb{R}),$$
$$\mathrm{Baexp}[\![A]\!](R)P \stackrel{\mathrm{def}}{=} \{\rho \in R \mid \exists i \in P \cap \mathbb{I} : \rho \vdash A \Mapsto i\} . (8)$$

$\forall P \in \wp(\mathbb{R}) : \lambda R \cdot \mathrm{Baexp}[\![A]\!](R)P$ is a lower closure operator

Structural definition of the backward/bottom-up collecting semantics of arithmetic expressions

$$\mathrm{Baexp}[\![\mathtt{n}]\!](R)P = (\underline{\mathtt{n}} \in P \cap \mathbb{I}\ ?\ R : \emptyset) \qquad (9)$$
$$\mathrm{Baexp}[\![\mathtt{X}]\!](R)P = \{\rho \in R \mid \rho(\mathtt{X}) \in P \cap \mathbb{I}\} \qquad (10)$$
$$\mathrm{Baexp}[\![\mathtt{?}]\!](R)P = (P \cap \mathbb{I} = \emptyset\ ?\ \emptyset : R) \qquad (11)$$
$$\mathrm{Baexp}[\![\mathtt{u}A']\!](R)P = \mathrm{Baexp}[\![A']\!](R)(\underline{\mathtt{u}}_{C}^{\triangleleft}(\mathrm{Faexp}[\![A']\!]R, P)) \qquad (12)$$
$$\text{where } \underline{\mathtt{u}}_{C}^{\triangleleft}(Q,P) \stackrel{\mathrm{def}}{=} \{v \in Q \mid \underline{\mathtt{u}}\, v \in P \cap \mathbb{I}\}$$
$$\mathrm{Baexp}[\![A_1\ \mathtt{b}\ A_2]\!](R)P = \qquad (13)$$
$$\text{let } \langle P_1,\ P_2\rangle = \underline{\mathtt{b}}_{C}^{\triangleleft}(\mathrm{Faexp}[\![A_1]\!]R, \mathrm{Faexp}[\![A_2]\!]R, P) \text{ in}$$
$$\mathrm{Baexp}[\![A_1]\!](R)P_1 \cap \mathrm{Baexp}[\![A_2]\!](R)P_2$$
$$\text{where } \underline{\mathtt{b}}_{C}^{\triangleleft}(P_1, P_2, P) \stackrel{\mathrm{def}}{=} \{\langle v_1,\ v_2\rangle \in P_1 \times P_2 \mid v_1\ \underline{\mathtt{b}}\ v_2 \in P \cap \mathbb{I}\}$$

Definition of the postcondition semantics of commands (recall from lecture 14)

The postcondition semantics of a command $C \in \mathrm{Com}$ (within a given program P) specifies the strongest postcondition $\mathrm{FPcom}[\![C]\!]R$ satisfied by environments resulting from the execution of the command C starting in any of the environments satisfying the precondition R, if and when this execution terminates.

$$\mathrm{FPcom} \in \mathrm{Com} \mapsto \wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathrm{Env}[\![P]\!]) \tag{14}$$
$$\mathrm{FPcom}[\![C]\!]R \stackrel{\mathrm{def}}{=} \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'\rangle\rangle \in \tau^\star[\![C]\!]\}$$

The postcondition semantics of a command can be understood, up to an interpretation, as a predicate transformer.

Backward/precondition collecting semantics

Definition of the backward/precondition collecting semantics

– The *backward collecting semantics* of commands C is

$$\mathrm{BPcom} \in \mathrm{Com} \mapsto \wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathrm{Env}[\![P]\!]) \tag{15}$$
$$\mathrm{BPcom}[\![C]\!]R \stackrel{\mathrm{def}}{=} \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'\rangle\rangle \in \tau^\star[\![C]\!]\}$$

– $\mathrm{BPcom}[\![C]\!]R$ is the weakest precondition for execution of P from entry point of command C to have the possibility to reach a final state in R on exit of the command

– The condition is necessary but not sufficient, because of non-determinism. So non-termination or termination in a state not in R is left opened

Structural specification of the backward collecting semantics

$$\mathrm{BPcom}[\![\mathtt{skip}]\!]R = R$$
$$\mathrm{BPcom}[\![\mathtt{X} := A]\!]R = \{\rho \mid \exists i \in \mathbb{I} : \rho \vdash A \Mapsto i \wedge \rho[\mathtt{X} := i] \in R\}$$
$$\mathrm{BPcom}[\![\mathtt{if}\ B\ \mathtt{then}\ S_t\ \mathtt{else}\ S_f\ \mathtt{fi}]\!]R =$$
$$\mathrm{Cbexp}[\![B]\!](\mathrm{BPcom}[\![S_t]\!]R) \cup \mathrm{Cbexp}[\![T(\neg(B))]\!](\mathrm{BPcom}[\![S_f]\!]R)$$
$$\mathrm{BPcom}[\![\mathtt{while}\ B\ \mathtt{do}\ S\ \mathtt{od}]\!]R =$$
$$\mathsf{lfp}_{\emptyset}^{\subseteq}\ \lambda X \cdot \mathrm{Cbexp}[\![T(\neg(B))]\!]R \cup \mathrm{Cbexp}[\![B]\!](\mathrm{BPcom}[\![S]\!]X)$$
$$\mathrm{BPcom}[\![C\ ;\ S]\!]R = (\mathrm{BPcom}[\![C]\!] \circ \mathrm{BPcom}[\![S]\!])R$$
$$\mathrm{BPcom}[\![S\ ;;]\!]R = \mathrm{BPcom}[\![S]\!]$$

Note

If we define:

$\mathbb{R}_P = \wp(\mathrm{Env}[\![P]\!])$

$\alpha_P[\![C]\!] \in \wp(\Sigma[\![P]\!] \times \Sigma[\![P]\!]) \longrightarrow (\wp(\mathbb{R}) \longrightarrow \wp(\mathbb{R}))$

$\alpha_P[\![C]\!]X = \lambda R.\{\rho \mid \exists \rho' \in R : \langle\langle at_P[\![C]\!], \rho\rangle, \langle after_P[\![C]\!], \rho'\rangle\rangle \in X\}$

then

$\mathcal{B}com[\![C]\!]R = \alpha[\![C]\!](\tau^*[\![C]\!])$ is an abstract interpretation of the big-step operational semantics $t^*[\![C]\!]$

□

Proof of correctness of the structural specification of the backward collecting semantics

PROOF.

- We observe that $\alpha_P[\![C]\!]$ is a complete join morphism:

$\alpha_P[\![C]\!](\bigcup_{i\in\Delta} X_i)$

$= \lambda R.\{\rho \mid \exists\rho'\in R : \langle\langle at_P[\![C]\!],\rho\rangle,\langle after_P[\![C]\!],\rho'\rangle\rangle \in \bigcup_{i\in\Delta} X_i\}$

$= \lambda R.\bigcup_{i\in\Delta}\{\rho \mid \exists\rho'\in R : \langle\langle at_P[\![C]\!],\rho\rangle,\langle after_P[\![C]\!],\rho'\rangle\rangle \in X_i\}$

$= \dot{\bigcup_{i\in\Delta}} \alpha_P[\![C]\!](X_i)$

so that this is the lower-adjoint of a Galois connection:

$\langle \wp(\Sigma[\![P]\!]\times\Sigma[\![P]\!]), \subseteq\rangle \underset{\alpha[\![C]\!]}{\overset{\gamma_P[\![C]\!]}{\leftrightarrows}} \langle \wp(\mathbb{R}) \to \wp(\mathbb{R}), \dot\subseteq\rangle$

for the pointwise extension $\dot\subseteq$ of $\subseteq$.

- We proceed by structural induction on the structure of programs.
- $\mathcal{B}com[\![skip]\!]R$

$\triangleq \alpha_P[\![skip]\!](t^*[\![skip]\!])R$

$= \alpha_P[\![skip]\!](1_{\Sigma[\![P]\!]} \cup \tau[\![skip]\!])R$

$= \{\rho \mid \exists\rho'\in R : \langle\langle at_P[\![skip]\!],\rho\rangle,\langle after_P[\![skip]\!],\rho'\rangle\rangle \in 1_{\Sigma[\![P]\!]}\cup\tau[\![skip]\!]\}$

$= \{\rho \mid \exists\rho'\in R : \rho'=\rho\}$ $\{at_P[\![C]\!] \neq after_P[\![C]\!]$ and def. $\tau[\![skip]\!]\}$

$= R$

- $\mathcal{B}com[\![X:=A]\!]R$

$\triangleq \alpha_P[\![X:=A]\!](t^*[\![X:=A]\!])R$

$= \alpha_P[\![X:=A]\!](1_{\Sigma[\![P]\!]}\cup\tau[\![X:=A]\!])R$

$= \{\rho\mid\exists\rho'\in R : \langle\langle at_P[\![X:=E]\!],\rho\rangle,\langle after_P[\![X:=E]\!],\rho'\rangle\rangle \in 1_{\Sigma[\![P]\!]}\cup\tau[\![X:=A]\!]\}$

$= \{\rho\mid\exists\rho'\in R : \langle\langle at_P[\![X:=A]\!],\rho\rangle,\langle after_P[\![X:=A]\!],\rho'\rangle\rangle \in \{\langle\langle at_P[\![X:=A]\!],\rho\rangle,\langle after_P[\![X:=A]\!],\rho[X\leftarrow i]\rangle\rangle \mid i\in\mathbb{I} \wedge \rho\vdash A \Rightarrow i\}\}$

$\{$since $at_P[\![C]\!]\neq after_P[\![C]\!]$ and def. $\tau[\![X:=A]\!]\}$

$= \{\rho\mid\exists\rho'\in R : \exists i\in\mathbb{I} : \rho'=\rho[X\leftarrow i]\wedge\rho\vdash A\Rightarrow i\}$

$= \{\rho\mid\exists i\in\mathbb{I} : \rho\vdash A\Rightarrow i \wedge \rho[X\leftarrow i]\in R\}$

- $\mathcal{B}\mathrm{com}[\![C]\!]R$ where $C = $ **if** B **then** S_t **else** S_f **fi**

$$\triangleq \alpha_p[\![C]\!]\,(t^*[\![C]\!])\,R$$

$$= \alpha_p[\![C]\!]\Big(\big((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ t^*[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)\big) \cup \big((1_{\Sigma[\![P]\!]} \cup \tau^{\overline{B}}) \circ t^*[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f)\big)\Big)\,R$$

$$= \alpha_p[\![C]\!]\big((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ t^*[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)\big)\,R$$

$$\dot{\cup}\ \alpha_p[\![C]\!]\big((1_{\Sigma[\![P]\!]} \cup \tau^{\overline{B}}) \circ t^*[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f)\big)\,R$$

We handle the case of the true alternative, the false alternative being handled in the same way.

$$\alpha_p[\![C]\!]\big((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ t^*[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)\big)\,R$$

$$= \{\rho \mid \exists \rho' \in R : \langle\!\langle \underline{\mathrm{at}}_p[\![C]\!], \rho\rangle, \langle \underline{\mathrm{after}}_p[\![C]\!], \rho'\rangle\!\rangle \in \big((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ t^*[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)\big)\}$$

⦃by def. $\alpha[\![C]\!]$⦄

$$= \{\rho \mid \exists \rho''' \in R : \exists \rho', \rho'' \in R : \exists \ell', \ell'' \in \underline{\mathrm{in}}_p[\![C]\!] :$$
$$\langle\!\langle \underline{\mathrm{at}}_p[\![C]\!], \rho\rangle, \langle \ell', \rho'\rangle\!\rangle \in (1_{\Sigma[\![P]\!]} \cup \tau^B)$$
$$\wedge\ \langle\!\langle \ell', \rho'\rangle, \langle \ell'', \rho''\rangle\!\rangle \in \tau^*[\![S_t]\!]$$
$$\wedge\ \langle\!\langle \ell'', \rho''\rangle, \langle \underline{\mathrm{after}}[\![C]\!], \rho'''\rangle\!\rangle \in (1_{\Sigma[\![P]\!]} \cup \tau^t)\}$$

⦃by def. of the composition $\circ$ of relations⦄

= ⦃We observe that $\underline{\mathrm{at}}_p[\![C]\!] = \ell'$ is impossible since $\ell' \in \underline{\mathrm{in}}_p[\![S_t]\!]$ would imply $\underline{\mathrm{at}}_p[\![C]\!] \in \underline{\mathrm{in}}_p[\![S_t]\!]$ in contradiction with (59) i.e. $\{\underline{\mathrm{at}}_p[\![C]\!], \underline{\mathrm{after}}_p[\![C]\!]\} \cap (\underline{\mathrm{in}}_p[\![S_t]\!] \cup \underline{\mathrm{in}}_p[\![S_f]\!]) = \emptyset$;
We observe that $\underline{\mathrm{after}}_p[\![C]\!] = \ell''$ is impossible for similar reasons.⦄

$$\{\rho \mid \exists \rho''' \in R : \exists \rho', \rho'' \in R : \exists \ell', \ell'' \in \underline{\mathrm{in}}_p[\![C]\!] :$$
$$\langle\!\langle \underline{\mathrm{at}}_p[\![C]\!], \rho\rangle, \langle \ell', \rho'\rangle\!\rangle \in \tau^B$$
$$\wedge\ \langle\!\langle \ell', \rho'\rangle, \langle \ell'', \rho''\rangle\!\rangle \in \tau^*[\![S_t]\!]$$
$$\wedge\ \langle\!\langle \ell'', \rho''\rangle, \langle \underline{\mathrm{after}}_p[\![C]\!], \rho'''\rangle\!\rangle \in \tau^t\}$$

= ⦃By def. of τ^B, $\rho' = \rho$ and $\ell' = \underline{\mathrm{at}}_p[\![S_t]\!]$. By def. of τ^t, $\rho'' = \rho'''$ and $\ell'' = \underline{\mathrm{after}}_p[\![S_t]\!]$⦄

$$\{\rho \mid \exists \rho''' \in R : \langle\!\langle \underline{\mathrm{at}}_p[\![C]\!], \rho\rangle, \langle \underline{\mathrm{at}}_p[\![S_t]\!], \rho\rangle\!\rangle \in \tau^B\ \wedge$$
$$\langle\!\langle \underline{\mathrm{at}}_p[\![S_t]\!], \rho\rangle, \langle \underline{\mathrm{after}}_p[\![S_t]\!], \rho'''\rangle\!\rangle \in \tau^*[\![S_t]\!]\ \wedge$$
$$\langle\!\langle \underline{\mathrm{after}}_p[\![S_t]\!], \rho'''\rangle, \langle \underline{\mathrm{after}}_p[\![C]\!], \rho'''\rangle\!\rangle \in \tau^t\}$$

= { by def. τ^B and τ^t }

$\{\rho \mid \exists \rho' \in R : \rho \vdash B \Rightarrow tt \wedge \langle\langle at_P[\![St]\!], \rho\rangle, \langle after[\![St]\!], \rho'\rangle\rangle \in \tau^*[\![St]\!]\}$

$= \{\rho \in \{\rho \mid \exists\rho'\in R : \langle\langle at_P[\![St]\!],\rho\rangle, \langle after[\![St]\!],\rho'\rangle\rangle \in \tau^*[\![St]\!]\}$
$\mid \rho\vdash B \Rightarrow tt\}$

$= \{\rho \in \alpha_P[\![St]\!](\tau^*[\![St]\!])R \mid \rho \vdash B \Rightarrow tt\}$

= { by def. $\mathcal{C}bexp[\![B]\!]R$ }

$\mathcal{C}bexp[\![B]\!](\alpha_P[\![St]\!](\tau^*[\![St]\!])R)$

= { by induction hypothesis }

$\mathcal{C}bexp[\![B]\!](\mathcal{B}com[\![St]\!]R)$

The false alternative is similar with Sf for St and $T(\neg B)$ for B.

– $\mathcal{B}com[\![C]\!]R$ where C = while B do S od

$= \alpha_P[\![C]\!](t^*[\![C]\!])R$

$= \alpha_P[\![C]\!](((1_{\Sigma[\![P]\!]} \cup t^*[\![S]\!]\circ\tau^R)\circ(\tau^B\circ\tau^*[\![S]\!]\circ\tau^R)^*\circ$
$(1_{\Sigma[\![P]\!]}\cup\tau^B\circ t^*[\![S]\!]\cup\tau^{\bar B})) \cup t^*[\![S]\!])R$

$= \{\rho \mid \exists\rho'\in R : \langle\langle at_P[\![C]\!],\rho\rangle, \langle after_P[\![C]\!],\rho'\rangle\rangle\in \tau^*[\![S]\!]\}$

$\cup \{\rho \mid \exists \rho'''\in R : \exists\rho',\rho''\in R : \exists \ell',\ell''\in in_P[\![C]\!] :$
$\langle\langle at_P[\![C]\!],\rho\rangle,\langle\ell',\rho'\rangle\rangle \in (1_{\Sigma[\![P]\!]}\cup t^*[\![S]\!]\circ\tau^R) \wedge$
$\langle\langle\ell',\rho'\rangle,\langle\ell'',\rho''\rangle\rangle\in(\tau^B\circ\tau^*[\![S]\!]\circ\tau^R)^*\wedge$
$\langle\langle\ell'',\rho''\rangle,\langle after_P[\![C]\!],\rho'''\rangle\rangle \in (1_{\Sigma[\![P]\!]}\cup\tau^B\circ t^*[\![S]\!]\cup\tau^{\bar B})\}$

= { The first term is $\emptyset$ since $\{at_P[\![C]\!], after_P[\![C]\!]\} \cap in_P[\![S]\!] = \emptyset$ by (60) and $\langle\langle\ell_1,\rho_1\rangle,\langle\ell_2,\rho_2\rangle\rangle \in \tau^*[\![S]\!]$ implies $\ell_1,\ell_2 \in in_P[\![S]\!]$.

. for the second term, $\ell' = at_P[\![C]\!]$ since otherwise $\exists \ell_2,\rho_2 :$ $\langle\langle at_P[\![C]\!],\rho\rangle,\langle\ell_2,\rho_2\rangle\rangle \in t^*[\![S]\!]\circ\tau^R$ which would imply $at_P[\![C]\!] \in in_P[\![S]\!]$ in contradiction with (60).

. for the second term, $\langle\langle\ell'',\rho''\rangle,\langle after_P[\![C]\!],\rho'''\rangle\rangle \in 1_{\Sigma[\![P]\!]}$ is impossible since then $\langle\langle\ell',\rho'\rangle,\langle after_P[\![C]\!],\rho''\rangle\rangle \in (\tau^B\circ\tau^*[\![S]\!]\circ\tau^R)^n$, with $n\geq 0$ which is itself impossible since either $n=0$ and $\ell' = after_P[\![C]\!]$ so $\ell' = at_P[\![C]\!] = after_P[\![C]\!]$ in contradiction with (56) or $n>0$ and $\exists \ell_1,\rho_1 : \langle\langle\ell_1,\rho_1\rangle,\langle after_P[\![C]\!],\rho''\rangle\rangle \in \tau^R$ in contradiction with the definition of τ^R and (56) }

$\{\rho \mid \exists\rho'''\in R : \langle\langle at_P[\![C]\!],\rho\rangle,\langle\ell'',\rho''\rangle\rangle \in (\tau^B\circ\tau^*[\![S]\!]\circ\tau^R)^*$
$\wedge \langle\langle\ell'',\rho''\rangle,\langle after_C[\![C]\!],\rho'''\rangle\rangle \in \tau^{\bar B}\}$

= { by def. $\tau^{\bar B}$ so that $\ell'' = at_P[\![C]\!]$ and $\rho''=\rho'''$ }

$\{\rho \mid \exists\rho'\in R : \langle\langle at_P[\![C]\!],\rho\rangle,\langle at_P[\![C]\!],\rho'\rangle\rangle \in (\tau^B\circ\tau^*[\![S]\!]\circ\tau^R)^*$
$\wedge\ \rho' \vdash T(\neg B) \Rightarrow tt\}$

$= \{\rho \mid \exists \rho' \in \{\rho'' \in \mathcal{R} \mid \rho'' \vdash T(\neg B) \Rightarrow tt\} : \langle\langle at_P[\![C]\!], \rho\rangle, \langle at_P[\![C]\!], \rho'\rangle\rangle \in (t^B \circ \tau^*[\![S]\!] \circ \tau^R)^*\}$

$= (\text{def. } \mathcal{C}bexp\,)$

$\{\rho \mid \exists \rho' \in \mathcal{C}bexp[\![T(\neg B)]\!]R : \langle\langle at_P[\![C]\!], \rho\rangle, \langle at_P[\![C]\!], \rho'\rangle\rangle \in (t^B \circ \tau^*[\![S]\!] \circ \tau^R)^*\}$

$= \alpha'((t^B \circ \tau^*[\![S]\!] \circ \tau^R)^*)$

by defining (for a given while command C and a given $R \in \wp(\mathcal{R})$):

$\overline{R} = \mathcal{C}bexp[\![T(\neg B)]\!]R$

$\alpha'(t) \triangleq \{\rho \mid \exists \rho' \in \overline{R} : \langle\langle at_P[\![C]\!], \rho\rangle, \langle at_P[\![C]\!], \rho'\rangle\rangle \in t\}$

We have

$\alpha'(\bigcup_{i \in \Delta} t_i) = \bigcup_{i \in \Delta} \alpha'(t_i)$

whence a Galois connection

$\langle \wp(\Sigma[\![P]\!] \times \Sigma[\![P]\!]), \subseteq\rangle \underset{\alpha'}{\overset{\gamma'}{\leftrightarrows}} \langle \wp(\mathcal{R}), \subseteq\rangle$

such that

$\alpha'(t^*)$ where $t = (\tau^B \circ \tau^*[\![S]\!] \circ \tau^R)$

$= \alpha'(\mathrm{lfp}\ \lambda X.\, 1 \cup t \circ X)$

$= \mathrm{lfp}\ F'$

where $\alpha'(1 \cup t \circ X) = F'(\alpha'(X))$.

i.e. $\alpha'(1) \cup \alpha'(t \circ X) = F'(\alpha'(X))$

since α' is a complete join morphism.

- $\alpha'(1)$

$= \{\rho' \mid \exists \rho \in \overline{R} : \langle\langle at_P[\![C]\!], \rho\rangle, \langle at_P[\![C]\!], \rho'\rangle\rangle \in 1_{\Sigma[\![P]\!]}\}$

$= \{\rho' \mid \exists \rho \in \overline{R} : \rho' = \rho\}$

$= \overline{R}$

- $\alpha'(t \circ X)$

$= \{\rho \mid \exists \rho' \in \overline{R} : \langle\langle at_P[\![C]\!], \rho\rangle, \langle at_P[\![C]\!], \rho'\rangle\rangle \in \tau^B \circ \tau^*[\![S]\!] \circ \tau^R \circ X\}$

$= (\text{def. composition } \circ\,)$

$\{\rho \mid \exists \rho'''' \in \overline{R} : \exists \rho', \rho'', \rho''' \in \mathcal{R} : \exists \ell', \ell'', \ell''', \ell'''' \in in_P[\![C]\!] :$

$\quad \langle\langle at_P[\![C]\!], \rho\rangle, \langle \ell', \rho'\rangle\rangle \in \tau^B$

$\quad \wedge \langle\langle \ell', \rho'\rangle, \langle \ell'', \rho''\rangle\rangle \in \tau^*[\![S]\!]$

$\quad \wedge \langle\langle \ell'', \rho''\rangle, \langle \ell''', \rho'''\rangle\rangle \in \tau^R$

$\quad \wedge \langle\langle \ell''', \rho'''\rangle, \langle at_P[\![C]\!], \rho''''\rangle\rangle \in X\}$

$=$ (by def of τ^B, we have $\ell' = at_P[\![S]\!]$, $\rho' = \rho$ and by definition of τ^R, we have $\ell'' = \underline{after}_P[\![S]\!]$, $\rho'' = \rho'''$ and $\ell''' = \underline{after}_P[\![C]\!]$)

$\{\rho \mid \exists \rho'''' \in \overline{R} : \exists \rho'' \in \mathcal{R} : \rho \vdash B \Rightarrow tt$

$\quad \wedge \langle\langle at_P[\![S]\!], \rho\rangle, \langle \underline{after}_P[\![S]\!], \rho''\rangle\rangle \in \tau^*[\![S]\!]$

$\quad \wedge \langle\langle \underline{after}_P[\![S]\!], \rho''\rangle, \langle at_P[\![C]\!], \rho''''\rangle\rangle \in X\}$

$$= \{\rho \in \{\rho \mid \exists \rho'''' \in \overline{R} : \exists \rho'' \in R : \langle\langle at_P[\![S]\!], \rho\rangle, \langle after_P[\![S]\!], \rho''\rangle \in \tau^*[\![S]\!] \wedge \langle\langle after_P[\![S]\!], \rho''\rangle, \langle at_P[\![C]\!], \rho''''\rangle \in X\} \mid \rho \vdash B \Rightarrow tt\}$$

$= \{$ by def of $\mathcal{C}bexp$ $\}$

$$\mathcal{C}bexp[\![B]\!](\{\rho \mid \exists \rho' \in \{\rho \mid \exists \rho' \in \overline{R} : \langle after[\![S]\!], \rho\rangle, \langle at_P[\![C]\!], \rho'\rangle\rangle \in X\} : \langle\langle at_P[\![S]\!], \rho\rangle, \langle after_P[\![S]\!], \rho'\rangle \in \tau^*[\![S]\!]\})$$

$= \{$ $after[\![S]\!] = at[\![C]\!]$ and def. α' $\}$

$$\mathcal{C}bexp[\![B]\!](\{\rho \mid \exists \rho' \in \alpha'(X) : \langle\langle at_P[\![S]\!], \rho\rangle, \langle after_P[\![S]\!], \rho'\rangle \in \tau^*[\![S]\!]\})$$

$= \{$ def. $\alpha_P[\![S]\!]$ $\}$

$$\mathcal{C}bexp[\![B]\!](\alpha_P[\![S]\!](\tau^*[\![S]\!])(\alpha'(X)))$$

$= \{$ structural induction hypothesis $\}$

$$\mathcal{C}bexp[\![B]\!](\mathcal{B}com[\![S]\!](\alpha'(X)))$$

So $F'(X) = \mathcal{C}bexp[\![T(\neg B)]\!]R \;\cup$
$\qquad \mathcal{C}bexp[\![B]\!](\mathcal{B}com[\![S]\!]X)$

and $\mathcal{B}com[\![C]\!]R = \mathrm{lfp}^{\subseteq}_{\emptyset} F'$

■ $\mathcal{B}com[\![C; S]\!]R$

$= \alpha[\![C;S]\!](t^*[\![C; S]\!])R$

$= \alpha[\![C;S]\!](t^*[\![C]\!] \circ t^*[\![S]\!])R$

$= \{\rho \mid \exists \rho' \in R : \langle\langle at_P[\![C;S]\!], \rho\rangle, \langle after_P[\![C;S]\!], \rho'\rangle\rangle \in t^*[\![C]\!] \circ t^*[\![S]\!]\}$

$= \{\rho \mid \exists \rho' \in R : \langle\langle at_P[\![C]\!], \rho\rangle, \langle after_P[\![S]\!], \rho'\rangle\rangle \in t^*[\![C]\!] \circ t^*[\![S]\!]\}$

$\{$ by (57) $\}$

$= \{\rho \mid \exists \rho'' \in R, \rho' \in R, \ell' \in in_P[\![C;S]\!] :$
$\quad \langle\langle at_P[\![C]\!], \rho\rangle, \langle \ell', \rho'\rangle\rangle \in t^*[\![C]\!]$
$\quad \wedge \langle\langle \ell', \rho'\rangle, \langle after_P[\![S]\!], \rho''\rangle\rangle \in t^*[\![S]\!]\}$

$= \{$ since $\ell' \in in_P[\![C]\!]$ and $\ell' \in in_P[\![S]\!]$, we must have $\ell' = after_P[\![C]\!] = at_P[\![S]\!]$ $\}$

$\{\rho \mid \exists \rho'' \in R : \rho' \in : \langle\langle at_P[\![C]\!], \rho\rangle, \langle after_P[\![C]\!], \rho'\rangle \in t^*[\![C]\!] \wedge$
$\quad \langle\langle at_P[\![S]\!], \rho'\rangle, \langle after_P[\![S]\!], \rho''\rangle \in t^*[\![S]\!]\}$

$= \{\rho \mid \exists \rho' \in \{\rho' \mid \exists \rho'' \in R : \langle\langle at_P[\![S]\!], \rho'\rangle, \langle after_P[\![S]\!], \rho''\rangle \in \tau^*[\![S]\!]\} :$
$\quad \langle\langle at_P[\![C]\!], \rho\rangle, \langle after_P[\![C]\!], \rho'\rangle\rangle \in t^*[\![C]\!]\}$

$= \mathcal{B}com[\![C]\!](\{\rho' \mid \exists \rho'' \in R : \langle\langle at_P[\![S]\!]\rho'\rangle, \langle after_P[\![S]\!], \rho''\rangle\rangle \in t^*[\![S]\!]\})$

$= \mathcal{B}com[\![C]\!](\mathcal{B}com[\![S]\!] R)$

$= (\mathcal{B}com[\![C]\!] \circ \mathcal{B}com[\![S]\!])(R)$

- $\mathcal{B}\mathrm{Com}[\![C\,;;]\!]R$
$= \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_{P;;}[\![C;;]\!], \rho\rangle, \langle \underline{\mathrm{after}}_{P;;}[\![C;;]\!], \rho'\rangle\rangle \in t^\star[\![C;;]\!]\}$
$= \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_{C;}[\![C]\!], \rho\rangle, \langle \mathrm{after}_{C;}[\![C]\!], \rho'\rangle\rangle \in t^\star[\![C]\!]\}$
$= \mathcal{B}\mathrm{Com}[\![C]\!]R$

$\square$

$\square$

Backward/precondition relational abstract semantics

Definition of the backward/precondition collecting semantics

- The *abstract backward/precondition semantics* $\mathrm{Abcom}[\![C]\!]R$ of command C in abstract environment R satisfies:

$$\mathrm{Abcom}[\![C]\!]R \mathrel{\dot{\sqsupseteq}} \alpha(\mathrm{BPcom}[\![C]\!](\gamma(R)))$$

<u>Note</u> : this is the case when

$$\langle(\wp(\mathbb{R}_P) \to \wp(\mathbb{R}_P)), \dot{\subseteq}\rangle \xrightleftharpoons[\alpha]{\gamma} \langle L, \sqsubseteq\rangle.$$

where $\mathbb{R}_P = \wp(\mathrm{Env}[\![P]\!])$. In absence of best approximation one would use a concretization function only where the soundness condition $\alpha \circ F \circ \gamma \sqsubseteq F^\#$ becomes $F \circ \gamma \sqsubseteq \gamma \circ F^\#$ for all concrete/abstract operations $F/F^\#$ $\square$

Structural specification of the backward collecting semantics

$$\mathrm{Abcom}[\![\mathtt{skip}]\!]R \;=\; R$$

$$\mathrm{Abcom}[\![\mathtt{X} := A]\!]R \;\dot{\sqsupseteq}\; \alpha(\{\rho \mid \exists i \in \mathbb{I} : \rho \vdash A \mapsto i \wedge \rho[\mathtt{X} := i] \in \gamma(R)\})$$

$$\mathrm{Abcom}[\![\mathtt{if}\ B\ \mathtt{then}\ S_t\ \mathtt{else}\ S_f\ \mathtt{fi}]\!]R =$$
$$\mathrm{Abexp}[\![B]\!](\mathrm{Abcom}[\![S_t]\!]R) \;\dot{\sqcup}\; \mathrm{Abexp}[\![T(\neg(B))]\!](\mathrm{Abcom}[\![S_f]\!]R)$$

$$\mathrm{Abcom}[\![\mathtt{while}\ B\ \mathtt{do}\ S\ \mathtt{od}]\!]R =$$
$$\mathsf{lfp}_{\emptyset}^{\subseteq}\ \lambda X \cdot \mathrm{Abexp}[\![T(\neg(B))]\!]R \;\dot{\sqcup}\; \mathrm{Abexp}[\![B]\!](\mathrm{Abcom}[\![S]\!]X)$$

$$\mathrm{Abcom}[\![C\ ;\ S]\!]R \;=\; (\mathrm{Abcom}[\![C]\!] \circ \mathrm{Abcom}[\![S]\!])R$$

$$\mathrm{Abcom}[\![S\ ;;]\!]R \;=\; \mathrm{Abcom}[\![S]\!]$$

PROOF. The soundness condition is:

$$\alpha \circ \mathrm{Bcom}[\![C]\!] \circ \gamma \sqsubseteq \mathrm{Abcom}[\![C]\!]$$

for Galois connections, or more generally

$$\mathrm{Bcom}[\![C]\!] \circ \gamma \sqsubseteq \mathrm{Abcom}[\![C]\!] \circ \gamma$$

in absence of best approximation and use of a concretization function only.

The proof is by structural induction on commands

- $\alpha \circ \mathrm{Bcom}[\![\mathtt{skip}]\!] \circ \gamma$
$= \alpha \circ \gamma$ {by def. $\mathrm{Bcom}[\![\mathtt{skip}]\!]$}
$\sqsubseteq 1$ {$\alpha \circ \gamma$ is reductive}.
$\triangleq \mathrm{Abcom}$

In case of concretization only we have
$\mathrm{Bcom}[\![\mathtt{skip}]\!] \circ \gamma \sqsubseteq \gamma \circ \mathrm{Abcom}[\![\mathtt{skip}]\!]$
$\Leftrightarrow 1 \circ \gamma \sqsubseteq \gamma \circ 1$ {where 1 is identity}
$\Leftrightarrow$ true

- For the assignment the soundness follows directly from its definition and that of Bcom.

- For the test $C =$ if Bexp then LC_1 else LC_2 fi

$(\alpha \circ \mathrm{Bcom}[\![C]\!] \circ \gamma)R$
$= \alpha(\mathrm{Cbexp}[\![\mathrm{Bexp}]\!](\mathrm{Bcom}[\![LC_1]\!]\gamma(R))$
$\cup\ \mathrm{Cbexp}[\![T(\neg \mathrm{Bexp})]\!](\mathrm{Bcom}[\![LC_2]\!]\gamma(R)))$
$=$ {$\gamma \circ \alpha$ is extensive, $\mathrm{Cbexp}[\![B]\!]$ is monotonic and α is a complete join morphism}

$$\alpha(\mathit{Cbexp}[\![\mathit{Bexp}]\!](\gamma(\alpha(\mathit{Bcom}[\![\mathit{LC}_1]\!]\,\gamma(R)))))$$
$$\sqcup\ \alpha(\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!](\gamma(\alpha(\mathit{Bcom}[\![\mathit{LC}_2]\!]\,\gamma(R)))))$$
$$\sqsubseteq \{\text{induction hypothesis}, \alpha, \mathit{Cbexp}[\![B]\!], \gamma \text{ monotone}\}$$
$$\alpha(\mathit{Cbexp}[\![\mathit{Bexp}]\!](\gamma(\mathit{Abcom}[\![\mathit{LC}_1]\!]R)))$$
$$\sqcup\ \alpha(\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!](\gamma(\mathit{Abcom}[\![\mathit{LC}_2]\!]R)))$$
$$\sqsubseteq \{\alpha\circ\mathit{Cbexp}[\![B]\!]\circ\gamma \mathrel{\dot{\sqsubseteq}} \mathit{Cbexp}^{\#}[\![B]\!] \text{ hypothesis}\}$$
$$\mathit{Cbexp}^{\#}[\![\mathit{Bexp}]\!](\mathit{Abcom}[\![\mathit{LC}_1]\!]R)$$
$$\sqcup\ \mathit{Cbexp}^{\#}[\![T(\neg\mathit{Bexp})]\!](\mathit{Abcom}[\![\mathit{LC}_2]\!]R)$$
$$\triangleq \mathit{Abcom}[\![C]\!]R.$$

In case of a γ-based abstract interpretation, we would have

$$(\mathit{Cbexp}[\![\mathit{Bexp}]\!](\mathit{Bcom}[\![\mathit{LC}_1]\!](\gamma(R))))$$
$$\cup\ (\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!](\mathit{Bcom}[\![\mathit{LC}_2]\!](\gamma(R))))$$
$$\subseteq \{\mathit{Cbexp}[\![B]\!] \text{ is monotone and induction hypothesis}\}$$
$$\mathit{Cbexp}[\![\mathit{Bexp}]\!](\gamma(\mathit{Abcom}[\![\mathit{LC}_1]\!]R))$$
$$\cup\ \mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!](\gamma(\mathit{Abcom}[\![\mathit{LC}_2]\!]R))$$
$$\subseteq \{\mathit{Cbexp}[\![B]\!]\circ\gamma \mathrel{\dot{\subseteq}} \gamma\circ\mathit{Cbexp}^{\#}[\![B]\!]\}$$
$$\gamma(\mathit{Cbexp}^{\#}[\![\mathit{Bexp}]\!](\mathit{Abcom}[\![\mathit{LC}_1]\!]R))$$
$$\cup\ \gamma(\mathit{Cbexp}^{\#}[\![T(\neg\mathit{Bexp})]\!](\mathit{Abcom}[\![\mathit{LC}_2]\!]R))$$

$$\subseteq \{\gamma \text{ is monotone so } \gamma(X)\cup\gamma(Y)\subseteq\gamma(X\sqcup Y) \text{ where } \sqcup \text{ is the abstract lub}\}$$
$$\gamma(\mathit{Cbexp}^{\#}[\![\mathit{Bexp}]\!](\mathit{Abcom}[\![\mathit{LC}_1]\!]R)\ \sqcup$$
$$\mathit{Cbexp}^{\#}[\![T(\neg\mathit{Bexp})]\!](\mathit{Abcom}[\![\mathit{LC}_2]\!]R))$$
$$= \gamma(\mathit{Abcom}[\![C]\!]R).$$

- For the while loop $C = \underline{\texttt{while}}\ \mathit{Bexp}\ \underline{\texttt{do}}\ \mathit{LC}\ \underline{\texttt{od}}$, we have:

$$\alpha(\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!]\gamma(R)\cup\mathit{Cbexp}[\![\mathit{Bexp}]\!](\mathit{Bcom}[\![\mathit{LC}]\!]X))$$
$$= \alpha(\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!]R)$$
$$\sqcup\ \alpha(\mathit{Cbexp}[\![\mathit{Bexp}]\!](\mathit{Bcom}[\![\mathit{LC}]\!]X))$$
$$\{\text{since } \alpha \text{ is a complete join morphism}\}$$
$$\sqsubseteq \alpha(\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!]\gamma(R))\ \sqcup$$
$$\alpha(\mathit{Cbexp}[\![\mathit{Bexp}]\!]\gamma(\alpha(\mathit{Bcom}[\![\mathit{LC}]\!]X)))$$
$$\{\text{since } \gamma\circ\alpha \text{ is extensive and } \mathit{Cbexp}[\![\mathit{Bexp}]\!] \text{ is monotone}\}$$
$$\sqsubseteq \alpha(\mathit{Cbexp}[\![T(\neg\mathit{Bexp})]\!]\gamma(R))\ \sqcup$$
$$\alpha(\mathit{Cbexp}[\![\mathit{Bexp}]\!]\gamma(\mathit{Abcom}[\![\mathit{LC}]\!](\alpha(X))))$$
$$\{\text{by induction hypothesis and monotony}\}$$
$$\sqsubseteq \mathit{Cbexp}^{\#}[\![T(\neg\mathit{Bexp})]\!]R\ \sqcup\ \mathit{Cbexp}^{\#}[\![\mathit{Bexp}]\!](\mathit{Abcom}[\![\mathit{LC}]\!](\alpha(X)))$$

so by the fixpoint approximation, theorem we conclude that

$$\alpha(\mathcal{B}com[\![\text{while } Bexp \text{ do } LCo \text{ od}]\!]\, \gamma(R))$$

$$= \alpha\Big(\mathrm{lfp}^{\subseteq}_{\emptyset}\, \lambda X.\, \mathcal{C}bexp[\![T(\neg Bexp)]\!]\, \gamma(R) \cup \mathcal{C}bexp[\![Bexp]\!](\mathcal{B}com[\![LCo]\!]\, X)\Big)$$

$$\sqsubseteq \mathrm{lfp}^{\sqsubseteq}_{\perp}\, \lambda X.\, \mathcal{C}bexp^{\#}[\![T(\neg Bexp)]\!]\, R \sqcup \mathcal{C}bexp^{\#}[\![Bexp]\!](\mathcal{A}bcom[\![LCo]\!]\, X)$$

The case of a γ-based abstraction is similar. We have:

$$\mathcal{C}bexp[\![T(\neg Bexp)]\!]\, \gamma(R) \cup \mathcal{C}bexp[\![Bexp]\!](\mathcal{B}com[\![LCo]\!]\, \gamma(X))$$

$\subseteq$ { $\mathcal{C}bexp[\![B]\!]$ is monotone, hypothesis on $\mathcal{C}bexp^{\#}$ and induction hypothesis }

$$\gamma(\mathcal{C}bexp^{\#}[\![Bexp]\!]\, R) \cup \mathcal{C}bexp[\![Bexp]\!](\gamma(\mathcal{A}com[\![LCo]\!]\, X))$$

$\subseteq$ { we use once again the hypothesis on $\mathcal{C}bexp^{\#}$ such that $\mathcal{C}bexp[\![B]\!] \circ \gamma \subseteq \gamma \circ \mathcal{C}bexp^{\#}[\![B]\!]$ }

$$\gamma(\mathcal{C}bexp^{\#}[\![Bexp]\!]\, R) \cup \gamma(\mathcal{C}bexp^{\#}[\![B]\!](\mathcal{A}com[\![LCo]\!]\, X))$$

$\subseteq$ { γ is monotone so $X \sqsubseteq X \sqcup Y$ implies $\gamma(X) \subseteq \gamma(X \sqcup Y)$ similarly $\gamma(Y) \subseteq \gamma(X \sqcup Y)$ whence $\gamma(X) \cup \gamma(Y) \subseteq \gamma(X \sqcup Y)$ }

$$\gamma(\mathcal{C}bexp^{\#}[\![Bexp]\!]\, R \sqcup \mathcal{C}bexp^{\#}[\![B]\!](\mathcal{A}com[\![LCo]\!]\, X))$$

Then we have $F \circ \gamma \subseteq \gamma \circ F^{\#}$ which implies $\mathrm{lfp}\, F \subseteq \gamma(\mathrm{lfp}\, F^{\#})$ since by transfinite induction $\perp \subseteq \gamma(\perp^{\#})$, if $X^{\delta} \subseteq \gamma(X^{\#\delta})$ then $F(X^{\delta}) \subseteq F \circ \gamma(X^{\#\delta}) \subseteq \gamma \circ F^{\#}(X^{\#\delta})$ so $X^{\delta+1} \subseteq \gamma(X^{\#\delta+1})$. For transfinite ordinals, $\forall X^{\beta} \subseteq \gamma(X^{\#\beta})$ for all $\beta < \lambda$, then $X^{\lambda} = \bigcup_{\beta<\lambda} X^{\beta} \subseteq \bigcup_{\beta<\lambda} \gamma(X^{\#\beta}) \subseteq \gamma(\bigsqcup_{\beta<\lambda} X^{\#\beta})$ since γ is monotonic whence $X^{\lambda} \subseteq \gamma(X^{\#\lambda})$. So $\mathrm{lfp}\, F = X^{\varepsilon} = X^{\max(\varepsilon,\varepsilon')} \subseteq X^{\#\max(\varepsilon,\varepsilon')} = X^{\#\varepsilon'} = \mathrm{lfp}^{\#} F$. We conclude that

$$\mathcal{B}com[\![\text{while } Bexp \text{ do } LCo \text{ od}]\!]\, \gamma(R)$$

$$= \mathrm{lfp}^{\subseteq}_{\emptyset}\, \lambda X.\, \mathcal{C}bexp[\![T(\neg Bexp)]\!]\, \gamma(R) \cup \mathcal{C}bexp[\![Bexp]\!](\mathcal{B}com[\![LCo]\!]\, X)$$

$$\subseteq \gamma\Big(\mathrm{lfp}^{\sqsubseteq}_{\perp}\, \lambda X.\, \mathcal{C}bexp^{\#}[\![T(\neg Bexp)]\!]\, R \sqcup \mathcal{C}bexp^{\#}[\![Bexp]\!](\mathcal{A}bcom[\![LCo]\!]\, X)\Big).$$

– We have

$$\alpha \circ \mathcal{B}com[\![Com\, ;\, LCo]\!] \circ \gamma$$

$$= \alpha \circ \mathcal{B}com[\![Com]\!] \circ \mathcal{B}com[\![LCo]\!] \circ \gamma$$

$\sqsubseteq$ { $\gamma \circ \alpha$ is extensive, α and $\mathcal{B}com[\![Com]\!]$ are monotonic }

$$\alpha \circ \mathcal{B}com[\![Com]\!] \circ \gamma \circ \alpha \circ \mathcal{B}com[\![LCo]\!] \circ \gamma$$

$\sqsubseteq$ { Induction hypothesis and monotony }

$\alpha \circ \mathrm{Bcom}[\![\mathrm{Com}]\!] \circ \gamma \circ \mathrm{Abcom}[\![\mathrm{LCo}]\!]$

$\sqsubseteq$ { induction hypothesis }

$\mathrm{Abcom}[\![\mathrm{Com}]\!] \circ \mathrm{Abcom}[\![\mathrm{LCo}]\!]$.

In the case of a γ-based abstraction, we have similarly,

$\mathrm{Bcom}[\![\mathrm{Com} ; \mathrm{LCo}]\!](\gamma(X))$

$= \mathrm{Bcom}[\![\mathrm{Com}]\!](\mathrm{Bcom}[\![\mathrm{LCo}]\!](\gamma(X)))$

$\sqsubseteq$ { monotony and ind. hyp. }

$\mathrm{Bcom}[\![\mathrm{Com}]\!](\gamma(\mathrm{Abcom}[\![\mathrm{CLo}]\!] X))$

$\sqsubseteq$ { ind. hyp. }

$\gamma(\mathrm{Abcom}[\![\mathrm{Com}]\!] \circ \mathrm{Abcom}[\![\mathrm{CLo}]\!](X))$

$= \gamma(\mathrm{Abcom}[\![\mathrm{Com} ; \mathrm{CLo}]\!] X)$.

- Finally $\alpha \circ \mathrm{Bcom}[\![\mathrm{LCo} ;;]\!] \circ \gamma = \alpha \circ \mathrm{Bcom}[\![\mathrm{LCo}]\!] \circ \gamma \sqsubseteq \mathrm{Abcom}[\![\mathrm{LCo}]\!] = \mathrm{Abcom}[\![\mathrm{LCo} ;;]\!]$. Similarly, $\mathrm{Bcom}[\![\mathrm{LCo} ;;]\!](\gamma(X)) = \mathrm{Bcom}[\![\mathrm{LCo}]\!]\gamma(X) \sqsubseteq \gamma(\mathrm{Abcom}[\![\mathrm{LCo}]\!]X)$ by induction hypothesis $= \gamma(\mathrm{Abcom}[\![\mathrm{LCo} ;;]\!]X)$.

$\square$

$\square$

- Note 1: The proof of the γ-based abstraction implies that of a Galois connection-based abstraction since $F \circ \gamma \sqsubseteq \gamma \circ F^\#$ implies $\alpha \circ F \circ \gamma \sqsubseteq \alpha \circ \gamma \circ F^\# \sqsubseteq F^\#$ (monotony, $\gamma \circ \alpha$ reductive)
- Note 2: We have never used the hypothesis that $\mathrm{Abcom}[\![B]\!]$ is monotonic so that the proof can be easily generalized to the widening/narrowing case.

Generic backward/precondition linear relational analyzer

Linear syntax

We just have to add a function `at` mapping linearized commands to their entry point:

```
(* linear_Syntax.mli *)
open Abstract_Syntax
(* A linear arithmetic expression a1.x1+...+an.xn+b, where n is the *)
(* number of program variables, is represented by a vector:         *)
(* LINEAR_AEXP a1 ... an b. A non-linear arithmetic expression is   *)
(* represented by RANDOM_AEXP.                                      *)
type laexp =
  | RANDOM_AEXP              (* random expression                   *)
  | LINEAR_AEXP of int array (* linear expression                   *)
and lbexp =
  | LTRUE | LFALSE     (* constant boolean expression               *)
  | RANDOM_BEXP        (* random boolean expression                 *)
  | LAND of lbexp list (* boolean conjunction                       *)
```



```
  | LOR  of lbexp list (* boolean disjunction                       *)
  | LGE  of int array  (* LGE a1 ... an b is a1.x1+...+an.xn >= b   *)
  | LEQ  of int array  (* LGE a1 ... an b is a1.x1+...+an.xn = b    *)
and label = Abstract_Syntax.label
and lcom =
  | LSKIP of label * label
  | LASSIGN of label * variable * laexp * label
  | LSEQ of label * (lcom list) * label
  | LIF of label * lbexp * lbexp * lcom * lcom * label
  | LWHILE of label * lbexp * lbexp * lcom * label
val at : lcom -> label            (* command entry label            *)
val after : lcom -> label         (* command exit label             *)
val incom : label -> lcom -> bool (* label in command               *)

(* linear_Syntax.mli *)
open Abstract_Syntax
(* A linear arithmetic expression a1.x1+...+an.xn+b, where n is the *)
(* number of program variables, is represented by a vector:         *)
```



```
(* LINEAR_AEXP a1 ... an b. A non-linear arithmetic expression is   *)
(* represented by RANDOM_AEXP.                                      *)
type laexp =
  | RANDOM_AEXP              (* random expression                   *)
  | LINEAR_AEXP of int array (* linear expression                   *)
and lbexp =
  | LTRUE | LFALSE     (* constant boolean expression               *)
  | RANDOM_BEXP        (* random boolean expression                 *)
  | LAND of lbexp list (* boolean conjunction                       *)
  | LOR  of lbexp list (* boolean disjunction                       *)
  | LGE  of int array  (* LGE a1 ... an b is a1.x1+...+an.xn >= b   *)
  | LEQ  of int array  (* LGE a1 ... an b is a1.x1+...+an.xn = b    *)
and label = Abstract_Syntax.label
and lcom =
  | LSKIP of label * label
  | LASSIGN of label * variable * laexp * label
  | LSEQ of label * (lcom list) * label
  | LIF of label * lbexp * lbexp * lcom * lcom * label
```



```
  | LWHILE of label * lbexp * lbexp * lcom * label
(* linearized command entry label *)
let at c = match c with
  | LSKIP (l,m)             -> l
  | LASSIGN (l,x,a,m)       -> l
  | LSEQ (l,s,m)            -> l
  | LIF (l,b,nb,st,sf,m)    -> l
  | LWHILE (l,b,nb,s,m)     -> l
(* linearized command exit label *)
let after c = match c with
  | LSKIP (l,m)             -> m
  | LASSIGN (l,x,a,m)       -> m
  | LSEQ (l,s,m)            -> m
  | LIF (l,b,nb,st,sf,m)    -> m
  | LWHILE (l,b,nb,s,m)     -> m
(* label in command     *)
let rec incom l c = match c with
  | LSKIP (l1,l2)           -> (l=l1) or (l=l2)
```

```
  | LASSIGN (l1,x,a,l2)    -> (l=l1) or (l=l2)
  | LSEQ (l1,s,l2)         -> (l=l1) or (l=l2) or (inseq l s)
  | LIF (l1,b,nb,st,sf,l2) -> (l=l1) or (l=l2) or (incom l st)
                                              or (incom l sf)
  | LWHILE (l1,b,nb,s,l2)  -> (l=l1) or (l=l2) or (incom l s)
and inseq l s = match s with
  | [] -> false
  | h::t -> (incom l h) or (inseq l t)
```


backward relational abstract interpretation of commands

```
(* bcom.mli *)
open Linear_Syntax
open Aenv
(* backward abstract interpretation of commands *)
val bcom : lcom -> Aenv.t -> label -> Aenv.t

(* bcom.ml *)
open Linear_Syntax
open Aenv
open Abexp
open Fixpoint
(* backward abstract semantics of commands *)
exception Error of string
```



```
let rec bcom c r l =
  match c with
  | (LSKIP (l', l'')) ->
      if (l = l') then r
      else if (l = l'') then r
      else (raise (Error "SKIP incoherence"))
  | (LASSIGN (l',x,a,l'')) ->
      if (l = l') then b_ASSIGN x a r
      else if (l = l'') then r
      else (raise (Error "ASSIGN incoherence"))
  | (LSEQ (l', s, l'')) ->
      (bcomseq s r l)
  | (LIF (l', b, nb, t, f, l'')) ->
      (if (l = l') then
         (join (a_bexp b (bcom t r (at t))) (a_bexp nb (bcom f r (at f))
      else if (incom l t) then
         (bcom t r l)
      else if (incom l f) then
```



```
         (bcom f r l)
      else if (l = l'') then
         r
      else (raise (Error "IF incoherence")))
  | (LWHILE (l', b, nb, c', l'')) ->
    let f x = (join (a_bexp b (bcom c' x (at c'))) (a_bexp nb r))
    in let i = lfp (bot ()) leq widen narrow f in
      (if (l = l') then i
       else if (incom l c') then (bcom c' i l)
       else if (l = l'') then r
       else (raise (Error "WHILE incoherence")))
and bcomseq s r l = match s with
  | []   -> raise (Error "empty SEQ incoherence")
  | [c]  -> if (incom l c) then (bcom c r l)
            else (raise (Error "SEQ incoherence"))
  | h::t -> if (incom l h) then (bcom h (bcomseq t r (at (List.hd t))) l
            else (bcomseq t r l)
```

The generic backward/precondition linear relational static analyzer

```
(* main.ml = main-bw.ml *)
open Program_To_Abstract_Syntax
open Labels
open Pretty_Print
open Lpretty_Print
open Abstract_To_Linear_Syntax
open Linear_Syntax
open Aenv
open Acom
open Bcom
let _ =
  let arg = if (Array.length  Sys.argv) = 1 then ""
            else Sys.argv.(1) in
```



```
     Random.self_init ();
    let p = (abstract_syntax_of_program arg) in
      (print_string "** Program:\n";
       pretty_print p;
       let p' = (linearize_com p) in
         print_string "** Linearized program:\n";
         lpretty_print p';
         init ();
         print_string "** Postcondition:\n";
         print (top ());
         print_string "** Precondition:\n";
         print (bcom p' (top ()) (at p'));
         quit ())
```


Polyhedral backward/precondition analysis

The polyhedral abstract environment for backward static analysis

– We just have to add the definition of `b_ASSIGN` to handle backward assignments:

```
(* aenv.mli *)
open Linear_Syntax
open Array
open Variables
(* set of environments *)
type t
(* relational library initialization *)
val init : unit -> unit
(* relational library exit *)
val quit : unit -> unit
(* infimum *)
```

```
val bot : unit -> t
(* check for infimum *)
val is_bot : t -> bool
(* uninitialization *)
val initerr : unit -> t
(* supremum *)
val top : unit -> t
(* least upper bound *)
val join : t -> t -> t
(* greatest lower bound *)
val meet : t -> t -> t
(* approximation ordering *)
val leq : t -> t -> bool
(* equality *)
val eq : t -> t -> bool
(* printing *)
val print : t -> unit
(* forward collecting semantics of assignment                   *)
```



```
(* f_ASSIGN x f r =  {e[x <- i] | e in r /\ i in f({e}) cap I } *)
val f_ASSIGN : variable -> laexp -> t -> t
(* backward collecting semantics of assignment                       *)
(* b_ASSIGN x f r =  {e | exists i in f({e}) cap I: e[x <- i] in r } *)
val b_ASSIGN : variable -> laexp -> t -> t
(* collecting semantics of boolean expressions                  *)
(* f_LGE a r = {e in r | a0.v0+...+an-1.vn-1 >= an             *)
val f_LGE : (int array) -> t -> t
(* f_LEQ a r = {e in r | a0.v0+...+an-1.vn-1 = an              *)
val f_LEQ : (int array) -> t -> t
(* convergence acceleration *)
(* widening *)
val widen : t -> t -> t
(* narrowing *)
val narrow : t -> t -> t

(* aenv.ml *)
open Linear_Syntax
```



```
open Variables
type lattice = BOT | TOP
type t =
  NULL of lattice  (* in absence of any variable, dimension = 0 *)
| POLY of Poly.t   (* must be of dimension > 0                  *)
exception PolyError of string
(* relational library initialization *)
let maxdims = 10000
let maxrows = 100
let init () = (Polka.initialize false maxdims maxrows;
               Polka.strict := false)
(* relational library exit (* print statistics *) *)
let quit () = Polka.finalize ()
(* infimum *)
let bot () = match (number_of_variables ()) with
 | 0 -> NULL BOT
 | n -> if (n < 0) then
     raise (PolyError "negative number of variables (bot)")
```



```
   else if (n > maxdims) then
     raise (PolyError "too many variables (bot)")
   else
     POLY (Poly.empty n) (* 1 <= n <= polka_maxcolumns-polka_dec *)
(* check for infimum *)
let is_bot r = match r with
 | NULL BOT -> true
 | NULL TOP -> false
 | POLY p -> (Poly.is_equal p (Poly.empty (number_of_variables ())))
(* uninitialization *)
let initerr () = match (number_of_variables ()) with
 | 0 -> NULL TOP
 | n -> if (n < 0) then
     raise (PolyError "negative number of variables (initerr)")
   else if (n > maxdims) then
     raise (PolyError "too many variables (initerr)")
   else
     POLY (Poly.universe n)
```

```
(* supremum *)
let top () =  match (number_of_variables ()) with
 | 0 -> NULL TOP
 | n -> if (n < 0) then
      raise (PolyError "negative number of variables (top)")
    else if (n > maxdims) then
      raise (PolyError "too many variables (top)")
    else
      POLY (Poly.universe n)
(* least upper bound *)
let ljoin l1 l2 = match (l1, l2) with
 | BOT, _ -> l2
 | _, BOT -> l1
 | _, _   -> TOP
let join r1 r2 = match (r1, r2) with
 | NULL l1, NULL l2 -> (NULL (ljoin l1 l2))
 | POLY p1, POLY p2 -> (POLY (Poly.union p1 p2))
 | _, _ -> raise (PolyError "join")
```



```
(* greatest lower bound *)
let lmeet l1 l2 = match (l1, l2) with
 | TOP, _ -> l2
 | _, TOP -> l1
 | _, _   -> BOT
let meet r1 r2 = match (r1, r2) with
 | NULL l1, NULL l2 -> (NULL (lmeet l1 l2))
 | POLY p1, POLY p2 -> (POLY (Poly.inter p1 p2))
 | _, _ -> raise (PolyError "meet")
(* approximation ordering *)
let lleq l1 l2 = match (l1, l2) with
 | BOT, _   -> true
 | _, TOP   -> true
 | TOP, BOT -> false
let leq r1 r2 = match (r1, r2) with
 | NULL l1, NULL l2 -> (lleq l1 l2)
 | POLY p1, POLY p2 -> (Poly.is_included_in p1 p2)
 | _, _ -> raise (PolyError "leq")
```



```
(* equality *)
let eq r1 r2 = match (r1, r2) with
 | NULL l1, NULL l2 -> (l1 = l2)
 | POLY p1, POLY p2 -> (Poly.is_equal p1 p2)
 | _, _ -> raise (PolyError "eq")
(* printing *)
let print r = match r with
 | NULL BOT -> (print_string "{ _|_ }\n")
 | NULL TOP -> (print_string "{ T }\n")
 | POLY p ->
   (Poly.minimize p; (* to get the constraints and generators of p *)
    Poly.print_constraints string_of_variable Format.std_formatter p;
    Format.pp_print_newline Format.std_formatter ())
(* convert a0.v0+...+an-1.vn-1+an where n = (number_of_variables ()) int
(* vector [1,an,a0,...,an-1].
let vector_of_lin_expr a =
  let v = Vector.make ((number_of_variables ()) + 2) in
    (Vector.set v 0 1;
```



```
     Vector.set v 1 (a.(number_of_variables ()));
     for i = 0 to ((number_of_variables ()) - 1) do
       Vector.set v (i+2) a.(i)
     done;
     (*
     Vector._print v;
     Vector.print_constraint string_of_variable Format.std_formatter v;
     Format.pp_print_newline Format.std_formatter ();
     *)
     v)
(* f_ASSIGN x f r =  {e[x <- i] | e in r /\ i in f({e}) cap I } *)
let f_ASSIGN x f r =
  match r with
 | NULL _ -> r
 | POLY p -> (match f with
 | RANDOM_AEXP ->
    let d = [|{ Polka.pos = x; Polka.nbdims = 1 }|] in
      (POLY (Poly.add_dims_and_embed_multi (Poly.del_dims_multi p d) d))
```

```
 | LINEAR_AEXP a ->
      (POLY (Poly.assign_var p x (vector_of_lin_expr a))))
(* b_ASSIGN x f r =  {e | exists i in f({e}) cap I: e[x <- i] in r } *)
let b_ASSIGN x f r =
  match r with
 | NULL _ -> r
 | POLY p -> POLY (match f with
 | RANDOM_AEXP ->
     let d = [|{ Polka.pos = x; Polka.nbdims = 1 }|] in
        (Poly.add_dims_and_embed_multi (Poly.del_dims_multi p d) d)
 | LINEAR_AEXP a ->
        (Poly.minimize p; (* <- to get around a bug of Polka 2.0.2   *)
                          (* in substitute_var                        *)
         Poly.substitute_var p x (vector_of_lin_expr a)))
(* f_LGE a r = {e in r | a0.v0+...+an-1.vn-1+an >= 0} *)
let f_LGE a r =
  match r with
 | NULL _ -> r
```



```
 | POLY p -> POLY (Poly.add_constraint p (vector_of_lin_expr a))
(* f_LEQ a r = {e in r | a0.v0+...+an-1.vn-1+an = 0} *)
let minus = Array.map (fun x -> (- x))
let f_LEQ a r = meet (f_LGE a r) (f_LGE (minus a) r)
(* widening *)
let widen r1 r2 = match (r1, r2) with
 | NULL l1, NULL l2 -> (NULL (ljoin l1 l2))
 | POLY p1, POLY p2 -> (POLY (Poly.widening p1 p2))
 | _, _ -> raise (PolyError "widen")
(* narrowing *)
(* let narrow a b = a (* does not ensure termination *) *)
let narrow a b = b (* less precise but does ensure termination *)
```


Examples of backward polyhedral static analysis

```
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out
x := 1;
while (x > 0) do
  x := x + 10000000
od;;
...
** Postcondition:
{1>=0}
** Precondition:
empty(1)
```

- The only way for the program execution to exit the loop is to *never start it* (because of arithmetic overflow when $\mathbb{I}$ is unbounded or non-termination if $\mathbb{N} \subseteq \mathbb{I}$)

The precondition/postcondition collecting semantics

Definition of the forward/backward semantics

- We assume that a transition system $\langle \Sigma,\ \tau,\ I,\ F\rangle$ is given such that
  - Σ is a set of states
  - $\tau \in \wp(\Sigma \times \Sigma)$ is the transition relation between a state and its potential successors
  - $I \subseteq \Sigma$ is the set of initial states
  - $F \subseteq \Sigma$ is the set of final states

- The forward or postcondition semantics is the set of states which are reachable from the initial states I

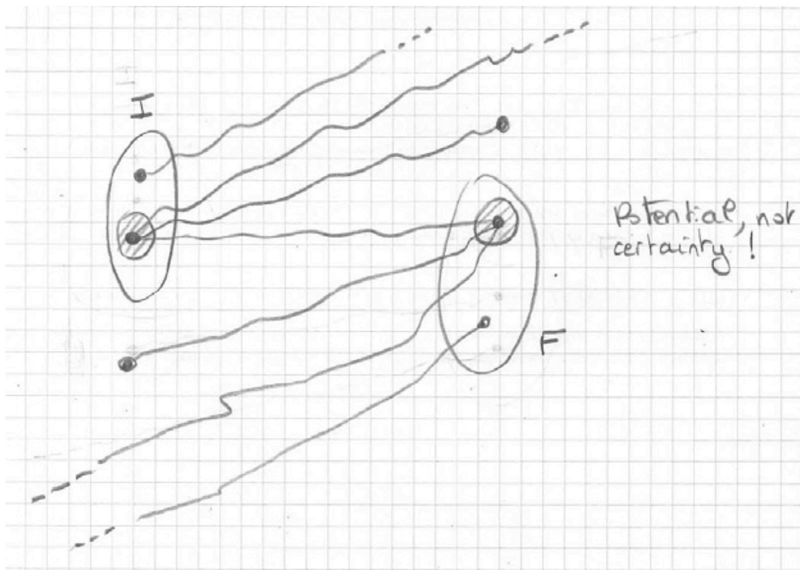
$$
\begin{aligned}
&\operatorname{post}[\tau^\star]I\\
&= \{s' \in \Sigma \mid \exists s \in I : \tau^\star(s, s')\}\\
&= \mathsf{lfp}^{\subseteq}_{\emptyset}\ \mathcal{F}
\end{aligned}
$$

$$
\begin{aligned}
\text{where}\quad & \mathcal{F}(X) = \ = I \cup \operatorname{post}[t]X\\
\text{and}\quad & \operatorname{post}[t]X = \{s' \in \Sigma \mid \exists s \in X : t(s, s')\}
\end{aligned}
$$

- The backward or precondition semantics is the set of states from which the final states F may be potentially reached

$$
\begin{aligned}
&\operatorname{pre}[\tau^\star]F\\
&= \{s \in \Sigma \mid \exists s' \in F : \tau^\star(s, s')\}\\
&= \mathsf{lfp}^{\subseteq}_{\emptyset}\ \mathcal{B}
\end{aligned}
$$

$$
\begin{aligned}
\text{where}\quad & \mathcal{B}(X) = F \cup \operatorname{pre}[t]X\\
\text{and}\quad & \operatorname{pre}[t]X = \{s \in \Sigma \mid \exists s' \in X : t(s, s')\}
\end{aligned}
$$



– The forward/backward or precondition/postcondition semantics is the pair

$$\langle I \cap \mathrm{pre}[\tau^\star]F,\ \mathrm{post}[\tau^\star]I \cap F\rangle$$

– In the concrete, there is nothing new

– However, in the abstract, we can do much better than separate overapproximations of $\mathrm{pre}[\tau^\star]F$ and $\mathrm{post}[\tau^\star]I$

Example of abstract forward/backward semantics

```
Generic-FW-BW-REL-Abstract-Interpreter % cat ../Examples/example48.sil
% example48.sil %
X := X + Y + 3;
while (X + Y <> 5)  do
  skip
od;;
Generic-FW-REL-Abstract-Interpreter % make pol
Polyhedral analysis
Generic-FW-REL-Abstract-Interpreter % ./a.out ../Examples/example48.sil
...
** Precondition:
{1>=0}
** Postcondition:
{X+Y=5}
```



```
Generic-FW-BW-REL-Abstract-Interpreter % make bw
Backward analysis
Generic-FW-BW-REL-Abstract-Interpreter % make pol
Polyhedral analysis
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example48.sil
** Postcondition:
{1>=0}
** Precondition:
{X+2Y=2}
Generic-FW-BW-REL-Abstract-Interpreter % make iter
Iterated forward/backward analysis
Generic-FW-BW-REL-Abstract-Interpreter % make pol
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example48.sil
** Precondition:
{X+2Y=2}
** Postcondition:
{X+Y=5}
```


Properties of the forward/backward collecting semantics

THEOREM. $\langle I \cap \mathrm{pre}[\tau^\star]F,\ F \cap \mathrm{post}[\tau^\star]I\rangle = \mathsf{gfp}^{\subseteq_2}\ \lambda\langle X,\ Y\rangle \cdot \langle I,\ F\rangle \cap_2 \langle X \cap \mathrm{pre}[\tau^\star]Y,\ Y \cap \mathrm{post}[\tau^\star]X\rangle$ ∎

PROOF. We prove that $\langle I \cap \mathrm{pre}[\tau^\star]F,\ F \cap \mathrm{post}[\tau^\star]I\rangle$ is a fixpoint of

$$\lambda\langle X,\ Y\rangle \cdot \langle I,\ F\rangle \cap_2 \langle X \cap \mathrm{pre}[\tau^\star]Y,\ Y \cap \mathrm{post}[\tau^\star]X\rangle$$

componentwise. For the first component, we have

$$\begin{aligned}
& I \cap (I \cap \mathrm{pre}[\tau^\star]F) \cap \mathrm{pre}[\tau^\star](F \cap \mathrm{post}[\tau^\star]I) \\
& \quad \text{(def. pre)} \\
= \ & I \cap \mathrm{pre}[\tau^\star]F \cap \{s \mid \exists s_2 : \tau^\star(s, s_2) \wedge s_2 \in F \wedge s_2 \in \mathrm{post}[\tau^\star]I\} \\
& \quad \text{(def. post)} \\
= \ & I \cap \mathrm{pre}[\tau^\star]F \cap \{s \mid \exists s_2 : \tau^\star(s, s_2) \wedge s_2 \in F \wedge \exists s_3 : s_3 \in I \wedge \tau^\star(s_3, s_2)\}
\end{aligned}$$

$\wr$def. pre$\wr$

$= I \cap \{s \mid \exists s_1 : s_1 \in F \wedge \tau^\star(s, s_A) \wedge \exists s_2 : \tau^\star(s, s_2) \wedge s_2 \in F \wedge \exists s_3 : s_3 \in I \wedge \tau^\star(s_3, s_2)\}$

$\wr$simplifying with $s_1 = s_2$ and $s_3 = s$$\wr$

$= I \cap \{s \mid \exists s_2 : \tau^\star(s, s_2) \wedge s_2 \in F\}$

$\wr$def. pre$\wr$

$= I \cap \mathrm{pre}[\tau^\star]F$

for I, F and τ.

To show that $\langle I \cap \mathrm{pre}[\tau^\star]F,\ F \cap \mathrm{post}[\tau^\star]I\rangle$ is the greatest fixpoint of

$$\lambda\langle X, Y\rangle \cdot \langle I, F\rangle \cap_2 \langle X \cap \mathrm{pre}[\tau^\star]Y,\ Y \cap \mathrm{post}[\tau^\star]X\rangle$$

we let $\langle X, Y\rangle$ be any such fixpoint. We have: $X = I \cap X \cap \mathrm{pre}[\tau^\star]Y$ so $X \subseteq I \cap \mathrm{pre}[\tau^\star]Y$. Moreover $Y = F \cap Y \cap \mathrm{post}[\tau^\star]X$ implies $Y \subseteq F$ so by monotony $X \subseteq I \cap \mathrm{pre}[\tau^\star]F$. Similarly, $Y \subseteq F \cap \mathrm{post}[\tau^\star]F$. $\square$

The precondition/postcondition abstract semantics and its implementation

Pre/postcondition abstract semantics

– We are given an abstraction $\langle \wp(\Sigma), \subseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle L, \sqsubseteq\rangle$

– We want to compute an overapproximation of:

$$\begin{aligned} &\langle \alpha(I \cap \mathrm{pre}[\tau^\star]F),\ \alpha(F \cap \mathrm{post}[\tau^\star]I)\rangle \\ =\ &\langle \alpha(I \cap \mathsf{lfp}\,\lambda Z \cdot F \cup \mathrm{pre}[\tau]Z),\ \alpha(F \cap \mathsf{lfp}\,\lambda Z \cdot I \cup \mathrm{post}[\tau]Z)\rangle \end{aligned}$$

– One solution is to overapproximate

$$\begin{aligned} \langle &\alpha(I) \sqcap \mathsf{lfp}\,\lambda Z \cdot \alpha(F) \sqcup \alpha(\mathrm{pre}[\tau](\gamma(Z)), \\ &\alpha(F) \sqcap \mathsf{lfp}\,\lambda Z \cdot \alpha(I) \sqcup \alpha(\mathrm{post}[\tau]\gamma(Z))\rangle \end{aligned}$$

– A better solution is to use the greatest fixpoint characterization and to overapproximate:

$$\begin{aligned} &\alpha_2(\mathsf{gfp}^{\subseteq_2}\,\lambda\langle X, Y\rangle \cdot \langle I, F\rangle \cap_2 \\ &\quad\langle X \cap \mathrm{pre}[\tau^\star]Y,\ Y \cap \mathrm{post}[\tau^\star]X\rangle) \\ &\quad\wr\text{fixpoint overapproximation}\wr \\ \sqsubseteq_2\ &\mathsf{gfp}^{\sqsubseteq_2}\,\lambda\langle X, Y\rangle \cdot \langle \alpha(I), \alpha(F)\rangle \sqcap_2 \\ &\quad\langle \alpha(\gamma(X) \cap \mathrm{pre}[\tau^\star]\gamma(Y)),\ \alpha(\gamma(Y) \cap \mathrm{post}[\tau^\star]\gamma(X))\rangle \\ &\quad\wr\text{fixpoint definition of } \mathrm{pre}[\tau^\star] \text{ and } \mathrm{post}[\tau^\star]\wr \\ =\ &\mathsf{gfp}^{\sqsubseteq_2}\,\lambda\langle X, Y\rangle \cdot \langle \alpha(I), \alpha(F)\rangle \sqcap_2 \\ &\quad\langle \alpha(\gamma(X) \cap \mathsf{lfp}\,\lambda Z \cdot \gamma(Y) \cup \mathrm{pre}[\tau]Z), \\ &\quad\ \alpha(\gamma(Y) \cap \mathsf{lfp}\,\lambda Z \cdot \gamma(X) \cup \mathrm{post}[\tau]Z)\rangle \end{aligned}$$

$\wr \alpha$ monotone $\wr$

$$\sqsubseteq_2 \ \mathsf{gfp}^{\sqsubseteq_2}\ \lambda\langle X,\ Y\rangle \cdot \langle \alpha(I),\ \alpha(F)\rangle \sqcap_2 \langle \alpha(\gamma(X)) \sqcap \mathsf{lfp}\ \lambda Z \cdot \alpha(\gamma(Y)) \sqcup \alpha(\mathrm{pre}[\tau](\gamma(Z))), \alpha(\gamma(Y)) \sqcap \lambda Z \cdot \alpha(\gamma(X)) \sqcup \alpha(\mathrm{post}[\tau]\gamma(Z))\rangle$$

$\wr \alpha \circ \gamma$ reductive and monotony $\wr$

$$\sqsubseteq_2 \ \mathsf{gfp}^{\sqsubseteq_2}\ \lambda\langle X,\ Y\rangle \cdot \langle \alpha(I),\ \alpha(F)\rangle \sqcap_2 \langle X \sqcap \mathsf{lfp}\ \lambda Z \cdot Y \sqcup \alpha(\mathrm{pre}[\tau](\gamma(Z))), Y \sqcap \lambda Z \cdot X \sqcup \alpha(\mathrm{post}[\tau]\gamma(Z))\rangle$$

Pre/postcondition static analysis

Assuming $\alpha(I) \sqsubseteq I^\sharp$, $\alpha(F) \sqsubseteq F^\sharp$, $\alpha \circ \mathrm{pre}[\tau] \circ \gamma \mathrel{\dot\sqsubseteq} \mathcal{B}^\sharp$, $\alpha \circ \mathrm{post}[\tau] \circ \gamma \mathrel{\dot\sqsubseteq} \mathcal{F}^\sharp$, and using chaotic iterations, the algorithm is

$$\begin{array}{rcl} X^0 & = & I^\sharp \\ Y^0 & = & F^\sharp \\ \ldots & \ldots & \ldots \\ X^{n+1} & = & X^n \sqcap \mathsf{lfp}\ \lambda Z \cdot Y^n \sqcup \mathcal{B}^\sharp(Z) \\ Y^{n+1} & = & Y^n \sqcap \mathsf{lfp}\ \lambda Z \cdot X^{n+1} \sqcup \mathcal{F}^\sharp(Z) \\ \ldots & \ldots & \ldots \end{array}$$

Convergence may have to be enforced via a narrowing.

Implementation of the pre/postcondition static analysis

In the implementation the initial precondition is that variables are uninitializated and the postcondition is $\mathtt{tt}$, that is $\top$ is both cases.

```
(* main.ml = main-iter-fw-bw.ml *)
open Program_To_Abstract_Syntax
open Labels
open Pretty_Print
open Lpretty_Print
open Abstract_To_Linear_Syntax
open Linear_Syntax
open Aenv
open Acom
open Bcom
```



```
let _ =
 let narrowing_limit = 100 in
  let arg = if (Array.length  Sys.argv) = 1 then ""
            else Sys.argv.(1) in
     Random.self_init ();
    let p = (abstract_syntax_of_program arg) in
      (print_string "** Program:\n";
       pretty_print p;
       let p' = (linearize_com p) in
         print_string "** Linearized program:\n";
         lpretty_print p';
         init ();
         let rec iterate pre n =
           (print_string "** Precondition:\n";
            print pre;
            let post = (acom p' pre (after p')) in
              (print_string "** Postcondition:\n";
               print post;
```

```
          let pre' = (bcom p' post (at p')) in
            (if (Aenv.eq pre pre') then
               (print_string "stable precondition after ";
                print_int n;print_string " iteration(s).\n")
             else if (n < narrowing_limit) then
               (print_string "unstable precondition after ";
                print_int n;print_string " iteration(s).\n";
                iterate pre' (n+1))
             else
               (print_string "stable stopped "; print_int n;
                print_string " iterations (narrowing).\n"))))
    in iterate (initerr ()) 1;
    quit ())
```


Examples of pre/postcondition static analysis

In the examples, we enforce a postcondition B as:

while $\neg B$ do skip od

which, since termination is enforced by the backward analysis, requires B to hold just before the loop.

```
Generic-FW-BW-REL-Abstract-Interpreter % make iter
Iterated forward/backward analysis
Generic-FW-BW-REL-Abstract-Interpreter % make pol
...
Polyhedral analysis
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example49.sil
```



```
** Program:
0:
  x := y;
1:
  z := ((2 * x) + 1);
2:
  while ((y < 0) | (0 < y)) do
    3:
      skip
    4:
  od {(y = 0)}
5:
```



```
** Linearized program:
0:
  x := 1.y + 0.x + 0.z + 0;
1:
  z := 0.y + 2.x + 0.z + 1;
2:
  while (-1.y + 0.x + 0.z + -1 >= 0 | 1.y + 0.x + 0.z + -1 >= 0) do
    3:
      skip
    4:
  od {-1.y + 0.x + 0.z + 0 = 0}
5:
```

```
** Precondition:
{1>=0}
** Postcondition:
{z=1,x=0,y=0}
unstable precondition after 1 iteration(s).
** Precondition:
{y=0}
** Postcondition:
{z=1,x=0,y=0}
stable precondition after 2 iteration(s).
Generic-FW-BW-REL-Abstract-Interpreter %
```

Comments on this example:

– The initial precondition {1>=0} is tt whence states no hypothesis on the input data

– The forward analysis provides the first postcondition stating that on termination y=0 whence z=1 and x=0

– Since y is not modified in the program, the next backward analysis provides the second precondition that we must have y=0 initially

– The last forward iteration shows that the fixpoint is reached

The following program is proved to never terminate:

```
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out
x := 1;
while (x > 0) do
  x := x + 1000000
od;;
** Precondition:
{1>=0}
** Postcondition:
empty(1)
unstable precondition after 1 iteration(s).
** Precondition:
empty(1)
** Postcondition:
empty(1)
stable precondition after 2 iteration(s).
```



```
Generic-FW-BW-REL-Abstract-Interpreter % cat ../Examples/example45.sil
% example45.sil %
X := X + Y;
while (X + Y <> 0)  do
  skip
od;;
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example45.sil
** Precondition:
{1>=0}
** Postcondition:
{X+Y=0}
unstable precondition after 1 iteration(s).
** Precondition:
{X+2Y=0}
** Postcondition:
{X+Y=0}
stable precondition after 2 iteration(s).
```

```
Generic-FW-BW-REL-Abstract-Interpreter % cat ../Examples/example46.sil
% example46.sil %
X := Y;
while (X + Y + Z <> 0)  do
  skip
od;;
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example46.sil
** Precondition:
{1>=0}
** Postcondition:
{2Y+Z=0,Y=X}
unstable precondition after 1 iteration(s).
** Precondition:
{2Y+Z=0}
** Postcondition:
{2Y+Z=0,Y=X}
stable precondition after 2 iteration(s).
```



```
Generic-FW-BW-REL-Abstract-Interpreter % cat ../Examples/example47.sil
% example47.sil %
X := Y;
while (X + Y + Z > 0)  do
  skip
od;;
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example47.sil
** Precondition:
{1>=0}
** Postcondition:
{Y=X,2Y+Z<=0}
unstable precondition after 1 iteration(s).
** Precondition:
{2Y+Z<=0}
** Postcondition:
{Y=X,2Y+Z<=0}
stable precondition after 2 iteration(s).
```

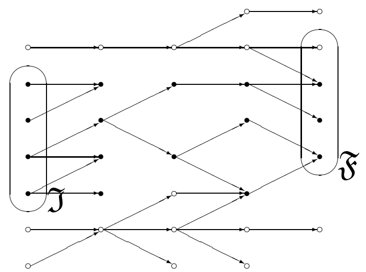

The forward/backward reachability collecting semantics

The forward reachability collecting semantics

– The forward reachability semantics

$$\text{post}[\tau^\star]I = \mathsf{lfp}^{\subseteq}\ \lambda X \cdot I \cup \text{post}[\tau]X$$

is the set of descendants of the initial states I by transitions τ



Recall the following fixpoint characterization of the forward reachability collecting semantics:

THEOREM. $\mathfrak{D} = \text{post}[\tau\star]\mathfrak{I} = \mathsf{lfp}\, F[\![\text{P}]\!]$ where $F[\![\text{P}]\!] \in \langle\wp(S),\, \cup\rangle \stackrel{\sqcup}{\longmapsto} \langle\wp(S),\, \cup\rangle$ is defined by $F[\![\text{P}]\!]X = \mathfrak{I} \cup \text{post}[\tau X$. ■

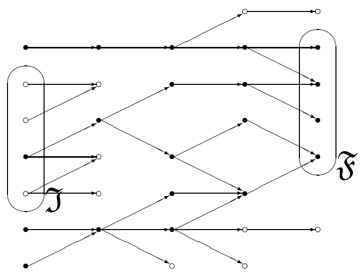
PROOF. Observe that $\wp(\mathfrak{S}\times\mathfrak{S})(\subseteq, \emptyset, \mathfrak{S}\times\mathfrak{S}, \cup, \cap)$ and $\wp(\mathfrak{S})(\subseteq, \emptyset, \mathfrak{S}, \cup, \cap)$ are complete lattices. Define $\alpha \in \wp(\mathfrak{S}\times\mathfrak{S}) \mapsto \wp(\mathfrak{S})$ by $\alpha(X) = \text{post}[X]\mathfrak{I}$. It is a complete $\cup$-morphism so that there exists γ such that $\wp(\mathfrak{S}\times\mathfrak{S})(\subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \wp(\mathfrak{S})(\subseteq)$. We have $\tau\star = \mathsf{lfp}\, T = \bigcup_{n\in\mathbb{N}} T^n(\emptyset)$ where $T(X) = 1 \cup X \circ \tau$, $\emptyset = \alpha(\emptyset)$ and $F[\![\text{P}]\!] \in \langle\wp(S),\, \cup\rangle \stackrel{\sqcup}{\longmapsto} \langle\wp(S),\, \cup\rangle$ is such that for all $X \in \wp(\mathfrak{S}\times\mathfrak{S})$, we have $\alpha \circ T(X) = \text{post}[T(X)]\mathfrak{I} = \{s \mid \exists s' \in \mathfrak{I} : \langle s',\, s\rangle \in T(X)\} = \{s \mid \exists s' \in \mathfrak{I} : \langle s',\, s\rangle \in 1 \cup X \circ \tau = \{s \mid \exists s' \in \mathfrak{I} : (\langle s',\, s\rangle \in 1) \vee (\langle s',\, s\rangle \in X \circ \tau\} = \{s \mid \exists s' \in \mathfrak{I} : (s' = s) \vee (\langle s',\, s\rangle \in X \circ \tau\} = \mathfrak{I} \cup \{s \mid \exists s' \in \mathfrak{I} : \langle s',\, s\rangle \in X \circ \tau = \mathfrak{I} \cup \{s \mid \exists s' \in \mathfrak{I} : \exists s'' \in \mathfrak{S} : \langle s',\, s''\rangle \in X \wedge s''\tau\} = \mathfrak{I} \cup \{s \mid \exists s'' \in \{s'' \mid \exists s' \in \mathfrak{I} : \langle s',\, s''\rangle \in X\} : s''\tau\} = \mathfrak{I} \cup \{s \mid \exists s'' \in \text{post}[X]\mathfrak{I} : s''\tau\} = \mathfrak{I} \cup \text{post}[\tau(\text{post}[X]\mathfrak{I}) = F[\![\text{P}]\!](\text{post}[X]\mathfrak{I}) = F[\![\text{P}]\!] \circ \alpha(X)$. By the fixpoint transfer theorem, we have $\alpha(\mathsf{lfp}\, T) = \mathsf{lfp}\, F[\![\text{P}]\!]$ so that $\mathfrak{D} = \text{post}[\tau\star]\mathfrak{I} = \alpha(\tau\star) = \alpha(\mathsf{lfp}\, T) = \mathsf{lfp}\, F[\![\text{P}]\!]$. □

The backward reachability collecting semantics

– The backward reachability semantics

$$\text{pre}[\tau^\star]F = \mathsf{lfp}^{\subseteq}\ \lambda Y \cdot F \cup \text{pre}[\tau]Y$$

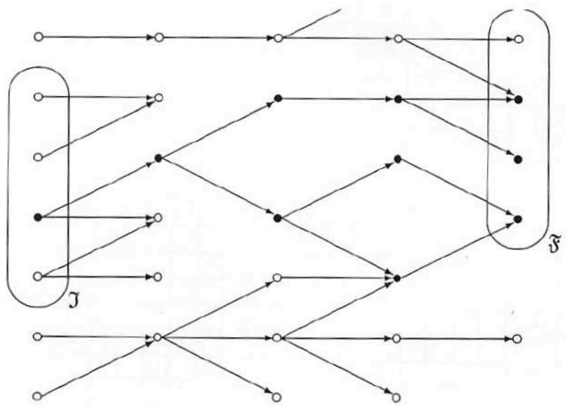
is the set of potential ascendants of the final states F by transitions τ



The forward/backward reachability collecting semantics

– We are interest in the forward/backward collecting semantics which is defined as $\text{post}[\tau^\star]I \cap \text{pre}[\tau^\star]F$ that is the intersection of two fixpoints $\mathsf{lfp}^{\subseteq}\ \lambda X \cdot I \cup \text{post}[\tau]X \cap \lambda Y \cdot F \cup \text{pre}[\tau]Y$

– We look for a more precise analysis than the mere intersection of the independent overapproximations of $\mathsf{lfp}^{\subseteq}\ \lambda X \cdot I \cup \text{post}[\tau]X$ and $\mathsf{lfp}^{\subseteq}\ \lambda Y \cdot F \cup \text{pre}[\tau]Y$



Descendant states (•) of the initial states (𝔍) which are ascendant states of the final states (𝔉).

Properties of the forward/backward collecting semantics

THEOREM. For all transition systems $\langle \Sigma, I, F, \tau\rangle$ where $\mathcal{F}(X) \stackrel{\text{def}}{=} I \cup \text{post}[\tau]X$, $\mathcal{B}(X) \stackrel{\text{def}}{=} I \cup \text{pre}[\tau]X$, and $X \subseteq \Sigma$, we have:

$$
\begin{aligned}
(\text{pre}[\tau]X) \cap \text{lfp}\,\mathcal{F} &\subseteq \text{pre}[\tau](X \cap \text{lfp}\,\mathcal{F}) &(16)\\
(\text{post}[\tau]X) \cap \text{lfp}\,B[\![\text{P}]\!] &\subseteq \text{post}[\tau](X \cap \text{lfp}\,\mathcal{B}) &(17)\\
\text{lfp}\,\mathcal{F} \cap \text{lfp}\,\mathcal{B} & &(18)\\
&= \text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{F} \cap \mathcal{B}(X)) &(19)\\
&= \text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{B} \cap \mathcal{F}(X)) &(20)\\
&= \text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{F} \cap \text{lfp}\,\mathcal{B} \cap \mathcal{B}(X)) &(21)\\
&= \text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{F} \cap \text{lfp}\,\mathcal{B} \cap \mathcal{F}(X)) &(22)
\end{aligned}
$$

■

PROOF. — To prove (16), observe that the theorem on page 113 implies that $(\text{pre}[\tau]X) \cap \text{lfp}\,\mathcal{F} = \{s \mid \exists s' \in X : s\tau s'\} \cap \{s \mid \exists s'' \in I : s''\tau^\star s\} \subseteq \{s \mid \exists s' \in X : \exists s'' \in I : s''\tau^\star s' \wedge s\tau s'\}$ since $s''\tau^\star s$ and $s\tau s'$ imply $s''\tau^\star s'$. This is precisely $\text{pre}[\tau](X \cap \text{post}[\tau^\star]I) = \text{pre}[\tau](X \cap \text{lfp}\,\mathcal{F})$. The proof of (17) is similar to that of (16).

— To prove (19), let $X^n, n \in \mathbb{N}$ and $Y^n, n \in \mathbb{N}$ be the iteration sequences starting from the infimum $\emptyset$ for $\mathcal{B}$ and $\lambda X \cdot (\text{lfp}\,\mathcal{F} \cap B[\![\,\text{P}]\!]X)$ respectively. We have $\text{lfp}\,\mathcal{F} \cap X^0 = \emptyset = Y^0$. Assume that $\text{lfp}\,\mathcal{F} \cap X^n \subseteq Y^n$ by induction hypothesis. Then $\text{lfp}\,\mathcal{F} \cap X^{n+1} = \text{lfp}\,\mathcal{F} \cap \mathcal{B}(X^n) = \text{lfp}\,\mathcal{F} \cap (\text{pre}[\tau]X^n \cup F) = \text{lfp}\,\mathcal{F} \cap \Big((\text{pre}[\tau]X^n \cap \text{lfp}\,\mathcal{F}) \cup F\Big)$, which, by (16), is included in $\text{lfp}\,\mathcal{F} \cap \Big(\text{pre}[\tau](X^n \cap \text{lfp}\,\mathcal{F}) \cup F\Big)$ which, by induction hypothesis and monotony, is included in $\text{lfp}\,\mathcal{F} \cap \big(\text{pre}[\tau](Y^n) \cup F\big) = \text{lfp}\,\mathcal{F} \cap \mathcal{B}Y^n = Y^{n+1}$. It follows that $\text{lfp}\,\mathcal{F} \cap \text{lfp}\,\mathcal{B} = (\cup_{n\in\mathbb{N}} X^n) \cap \text{lfp}\,\mathcal{F} = \cup_{n\in\mathbb{N}} (X^n \cap \text{lfp}\,\mathcal{F}) \subseteq \cup_{n\in\mathbb{N}} Y^n = \text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{F} \cap \mathcal{B}(X))$. But lfp is monotone so that $\text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{F} \cap \mathcal{B}(X)) \subseteq \text{lfp}\,\lambda X \cdot (\text{lfp}\,\mathcal{F}) \cap \text{lfp}\,\lambda X \cdot (\mathcal{B}(X)) = \text{lfp}\,\mathcal{F} \cap \text{lfp}\,\mathcal{B}$. Equality follows by antisymmetry. The proofs of (20) to (22) are similar. □

The forward/backward reachability abstract semantics

Let us recall the following fixpoint approximation result:

THEOREM. If $P(\preceq, f, t, \wedge, \vee)$ is a complete lattice, F, $\bar{F} \in P(\preceq) \stackrel{m}{\longmapsto} P(\preceq)$, and $F \preceq \bar{F}$, then $\text{lfp}\,F \preceq \text{lfp}\,\bar{F}$. ■

PROOF. We have $F(\text{lfp}\,\bar{F}) \preceq \bar{F}(\text{lfp}\,\bar{F}) = \text{lfp}\,\bar{F}$ whence $\text{lfp}\,F \preceq \text{lfp}\,\bar{F}$ since $\text{lfp}\,F = \bigwedge\{X \mid F(X) \preceq X\}$ by Tarski's fixpoint theorem. □

as well as fixpoint abstraction:

THEOREM. If $P(\preceq, f, t, \wedge, \vee)$ and $P^\sharp(\preceq^\sharp, f^\sharp, t^\sharp, \wedge^\sharp, \vee^\sharp)$ are complete lattices, $P(\preceq) \underset{\alpha}{\overset{\gamma}{\longleftrightarrow}} P^\sharp(\preceq^\sharp)$ and $F \in P(\preceq) \stackrel{m}{\longmapsto} P(\preceq)$, then $\alpha(\text{lfp}\,F) \preceq^\sharp \text{lfp}\,\alpha \circ F \circ \gamma$. ■

Proof. By Tarski's fixpoint theorem, the least fixpoints exist. So let $p^\sharp = \mathsf{lfp}\, \alpha \circ F \circ \gamma$. We have $\alpha \circ F \circ \gamma(p^\sharp) = p^\sharp$ whence $F \circ \gamma(p^\sharp) \preceq \gamma(p^\sharp)$ by def. Galois connection. It follows that $\gamma(p^\sharp)$ is a postfixpoint of F whence $\mathsf{lfp}\, F \preceq \gamma(p^\sharp)$ by Tarski's fixpoint theorem or equivalently $\alpha(\mathsf{lfp}\, F) \preceq^\sharp p^\sharp = \mathsf{lfp}\, \alpha \circ F \circ \gamma$. □

- In order to approximate $\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B$ from above using abstract interpretations $F^\sharp$ of F and $B^\sharp$ of B, we can use the abstract upper approximation $\mathsf{lfp}\, F^\sharp \wedge^\sharp \mathsf{lfp}\, B^\sharp$.
- However, a better approximation suggested in [2] can be obtained as the limit of the decreasing chain

$$\begin{aligned}
\dot{X}^0 &= \mathsf{lfp}\, F^\sharp \\
\dot{X}^{2n+1} &= \mathsf{lfp}\, \lambda X \cdot \dot{X}^{2n} \wedge^\sharp F^\sharp(X) \\
\dot{X}^{2n+2} &= \mathsf{lfp}\, \lambda X \cdot \dot{X}^{2n+1} \wedge^\sharp B^\sharp(X)
\end{aligned}$$

for all $n \in \mathbb{N}$.

- Observe that by theorem on page **117** there is no improvement when considering the exact collecting semantics.
- However, when considering approximations of the collecting semantics, not all information can be collected in one pass and iteration can lead definite improvement

- If the abstract lattice does not satisfy the descending chain condition then [2] also suggests to use a narrowing operator Δ to enforce convergence of the downward iteration $\dot{X}^k$, $k \in \mathbb{N}$.
- The same way a widening/narrowing approach can be used to enforce convergence of the iterates for $\lambda X \cdot \dot{X}^{2n} \wedge^\sharp F^\sharp(X)$ and $\lambda X \cdot \dot{X}^{2n+1} \wedge^\sharp B^\sharp(X)$.
- The correctness of this approach follows from the following result on fixpoint meet approximation:

Theorem. If $P(\preceq, f, t, \wedge, \vee)$ and $P^\sharp(\preceq^\sharp, f^\sharp, t^\sharp, \wedge^\sharp, \vee^\sharp)$ are complete lattices, $P(\preceq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} P^\sharp(\preceq^\sharp)$, $F \in P(\preceq) \xrightarrow{m} P(\preceq)$ and $B \in P(\preceq) \xrightarrow{m} P(\preceq)$ satisfy hypotheses (21) and (22) of theorem , $F^\sharp \in P^\sharp(\preceq^\sharp) \xrightarrow{m} P^\sharp(\preceq^\sharp)$, $B^\sharp \in P^\sharp(\preceq^\sharp) \xrightarrow{m} P^\sharp(\preceq^\sharp)$, $\alpha \circ F \circ \gamma \preceq^\sharp F^\sharp$, $\alpha \circ B \circ \gamma \preceq^\sharp B^\sharp$, $\dot{X}^0$ is $\mathsf{lfp}\, F^\sharp$ or $\mathsf{lfp}\, B^\sharp$ and for all $n \in \mathbb{N}$, $\dot{X}^{2n+1} = \mathsf{lfp}\, \lambda X \cdot (\dot{X}^{2n} \wedge^\sharp B^\sharp(X))$ and $\dot{X}^{2n+2} = \mathsf{lfp}\, \lambda X \cdot (\dot{X}^{2n+1} \wedge^\sharp F^\sharp(X))$ then $\forall k \in \mathbb{N} : \alpha(\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B) \preceq^\sharp \dot{X}^{k+1} \preceq^\sharp \dot{X}^k$. ■

Proof. Observe that by the fixpoint property, $\dot{X}^{2n+1} = \dot{X}^{2n} \wedge^\sharp B^\sharp(\dot{X}^{2n+1})$ and $\dot{X}^{2n+2} = \dot{X}^{2n+1} \wedge^\sharp F^\sharp(\dot{X}^{2n+2})$, hence $\dot{X}^{2n} \preceq^\sharp \dot{X}^{2n+1} \preceq^\sharp \dot{X}^{2n+2}$ since $\wedge^\sharp$ is the greatest lower bound for $\preceq^\sharp$ so that $\dot{X}^k$, $k \in \mathbb{N}$ is a decreasing chain.

We have $\alpha(\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B) \preceq^\sharp \alpha(\mathsf{lfp}\, F)$ since α is monotone and $\alpha(\mathsf{lfp}\, F) \preceq^\sharp \mathsf{lfp}\, F^\sharp$ by theorems on pages 120 and 120, thus proving the proposition for $k = 0$. Let us observe that $\alpha \circ F \circ \gamma \preceq^\sharp F^\sharp$ implies $F \circ \gamma \preceq \gamma \circ F^\sharp$ by def. Galois connection so that in particular for an argument of the form $\alpha(X)$, $F \circ \gamma \circ \alpha \preceq \gamma \circ F^\sharp \circ \alpha$. $\gamma \circ \alpha$ is extensive so that by monotony and transitivity $F \preceq \gamma \circ F^\sharp \circ \alpha$. Assume now by induction hypothesis that $\alpha(\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B) \preceq^\sharp \dot{X}^{2n}$, whence $\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B \preceq^\sharp \gamma(\dot{X}^{2n})$ by def. Galois connection. Since $F \preceq \gamma \circ F^\sharp \circ \alpha$, it follows that $\lambda X \cdot \mathsf{lfp}\, F \wedge \mathsf{lfp}\, B \wedge F(X) \preceq^\sharp \lambda X \cdot \gamma(\dot{X}^{2n}) \wedge \gamma \circ F^\sharp \circ \alpha(X) = \lambda X \cdot \gamma(\dot{X}^{2n} \wedge F^\sharp \circ \alpha(X))$ since γ is a complete meet morphism. Now by hypothesis (21) of theorel , we have $\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B = \mathsf{lfp}\, \lambda X \cdot (\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B \wedge F(X)) \preceq^\sharp \mathsf{lfp}\, \lambda X \cdot \gamma(\dot{X}^{2n} \wedge F^\sharp \circ \alpha(X))$ by theorem on page 120. Let G be $\lambda X \cdot \dot{X}^{2n} \wedge F^\sharp(X)$. $\alpha \circ \gamma$ is reductive so that by monotony $G \circ \alpha \circ \gamma \preceq^\sharp G$ and $\alpha \circ \gamma \circ G \circ \alpha \circ \gamma \preceq^\sharp G \circ \alpha \circ \gamma$, whence, by transitivity, $\alpha \circ \gamma \circ G \circ \alpha \circ \gamma \preceq^\sharp G$. By the theorem on page 120 we have $\alpha(\mathsf{lfp}\, \gamma \circ G \circ \alpha) \preceq^\sharp \mathsf{lfp}\, \alpha \circ \gamma \circ G \circ \alpha \circ \gamma \preceq^\sharp \mathsf{lfp}\, G$ by theorem on page 120. Hence, $\mathsf{lfp}\, \lambda X \cdot \gamma(\dot{X}^{2n} \wedge F^\sharp \circ \alpha(X)) \preceq \gamma(\mathsf{lfp}\, \lambda X \cdot \dot{X}^{2n} \wedge F^\sharp(X))$ so that by transitivity we conclude that $\alpha(\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B) \preceq^\sharp \dot{X}^{2n+1}$. The proof that $\alpha(\mathsf{lfp}\, F \wedge \mathsf{lfp}\, B) \preceq^\sharp \dot{X}^{2n+2}$ is similar using hypothesis (22) of theorem . $\square$

Example

– Let us consider the original example given in [2], with interval analysis, so that a widening/narrowing is required:

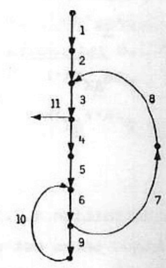
$$
\begin{array}{lll}
P^1 \sqsupseteq lfp(\overline{F}(\overline{\phi})) & & \\
P^2 = P^1 \underline{\Delta} X^2 & \text{où } X^2 & \sqsupseteq lfp(\lambda X . P^1 \sqcap \overline{B}(\overline{\Psi})(X)) \\
\ldots & & \\
P^{2k+1} = P^{2k} \underline{\Delta} X^{2k+1} & \text{où } X^{2k+1} & \sqsupseteq lfp(\lambda X . P^{2k} \sqcap \overline{F}(\overline{\phi})(X)) \\
P^{2k+2} = P^{2k+1} \underline{\Delta} X^{2k+2} & \text{où } X^{2k+2} & \sqsupseteq lfp(\lambda X . P^{2k+1} \sqcap \overline{B}(\overline{\phi})(X)) \\
\ldots & &
\end{array}
$$

– We illustrate on bubble sort [Z. Manna, 1974, p. 191]


```
{1}
        i:=n;
{2}
    L:
{3}
        si i≠0 alors
{4}
            j:=0;
{5}
    M:
{6}
            si j=i alors
{7}
                i:=i-1;
{8}
                allera L;
            finsi;
{9}
            j:=j+1;
{10}
            allera M;
        finsi;
{11}
```

– The analysis in [2] is not structural but equational and so the forward system of equations is:

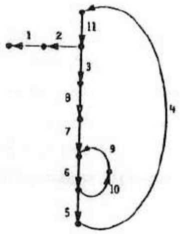
$$
\left\{
\begin{array}{lll}
P_1 & = & \overline{\phi} \\
P_2 & = & P_1(i \leftarrow n) \\
P_3 & = & P_2 \sqcup P_8 \\
P_4 & = & P_3(i \leftarrow i \neq 0) \\
P_5 & = & P_4(j \leftarrow [0,0]) \\
P_6 & = & P_5 \sqcup P_{10} \\
P_7 & = & P_6(i \leftarrow i \sqcap j, j \leftarrow i \sqcap j) \\
P_8 & = & P_7(i \leftarrow i-1) \\
P_9 & = & P_6(i \leftarrow i \neq j, j \leftarrow i \neq j) \\
P_{10} & = & P_9(j \leftarrow j+1) \\
P_{11} & = & P_3(i \leftarrow i \sqcap [0,0])
\end{array}
\right.
$$



Graphe de dépendance
(tête de cycle: 6)

– The backward system of equations is:

$$
\begin{cases}
P_1 = P_2(i \leftarrow [-\infty,+\infty], n \leftarrow i) \\
P_2 = P_3 \\
P_3 = P_{11} \sqcup P_4 \\
P_4 = P_5(j \leftarrow [-\infty,+\infty]) \\
P_5 = P_6 \\
P_6 = P_7 \sqcup P_9 \\
P_7 = P_8(i \leftarrow i+1) \\
P_8 = P_3 \\
P_9 = P_{10}(j \leftarrow j-1) \\
P_{10} = P_6 \\
P_{11} = \overline{\Psi}
\end{cases}
$$



Graphe de dépendance
(tête de cycle: 6)

– The considered program specification is:

$\overline{\Phi} = ((i=\bot),(j=\bot),(n=[-\infty,+\infty]))$
$\overline{\Psi} = ((i=[-\infty,+\infty]),(j=[-\infty,+\infty]),(n=[-\infty,+\infty]))$

– The first forward iteration is:

$P^1 \sqsupseteq \mathit{lfp}(\overline{F}(\overline{\phi}))$

	1	2	3	4	5	6	7	8	9	10	11
i	⊥	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[0,+∞]	[-1,+∞]	[-∞,+∞]	[-∞,+∞]	[0,0]
j	⊥	⊥	[0,+∞]	[0,+∞]	[0,0]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[1,+∞]	[0,+∞]
n	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]

It is discovered that j being initialized to 0 and incremented, it should be positive.

– The second iteration is:

$P^2 = P^1 \underline{\underline{\Delta}} X^2$ où $X^2 \sqsupseteq \mathit{lfp}(\lambda X. P^1 \sqcap \overline{B}(\overline{\Psi})(X))$

	1	2	3	4	5	6	7	8	9	10	11
i	⊥	[0,+∞]	[0,+∞]	[1,+∞]	[1,+∞]	[1,+∞]	[1,+∞]	[0,+∞]	[1,+∞]	[1,+∞]	[0,0]
j	⊥	⊥	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[1,+∞]	[0,+∞]
n	[0,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]	[-∞,+∞]

Since i is decremented by 1 in the loop which ends with i = 0, i must necessarily be positive on loop entry, so n must be initially positive

– The third iteration is:

$P^3 = P^2 \underline{\underline{\Delta}} X^3$ où $X^3 \sqsupseteq \mathit{lfp}(\lambda X. P^2 \sqcap \overline{B}(\overline{\Psi})(X))$

	1	2	3	4	5	6	7	8	9	10	11
i	⊥	[0,+∞]	[0,+∞]	[1,+∞]	[1,+∞]	[1,+∞]	[1,+∞]	[0,+∞]	[1,+∞]	[1,+∞]	[0,0]
j	⊥	⊥	[1,+∞]	[1,+∞]	[0,0]	[0,+∞]	[1,+∞]	[1,+∞]	[0,+∞]	[1,+∞]	[1,+∞]
n	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]	[0,+∞]

This propagates forward the information on n

– The next iteration shows that a fixpoint is reached

– The analysis proves that that if n is initially negative then the program exiecution either terminates by an error or does not terminate

– By propagating forward the invariants, one determines where execution tests have to be placed

– With polyhedral analysis, the result is the following:

```
Generic-FW-BW-REL-Abstract-Interpreter % cat ../Examples/example30.sil
i := n;
while (i <> 1) do
 j := 0;
 while(j <> i) do
   j := j + 1
 od;
 i := i - 1
od;;
```

```
Generic-FW-BW-REL-Abstract-Interpreter % ./a.out ../Examples/example30.sil
** Precondition:
{1>=0}
** Postcondition:
{i=1,n>=1}
unstable precondition after 1 iteration(s).
** Precondition:
{n>=1}
** Postcondition:
{i=1,n>=1}
stable precondition after 2 iteration(s).
```


Optimality

Lemma 1 *Let $P^\flat(\sqsubseteq^\flat, \bot^\flat, \top^\flat, \sqcap^\flat, \sqcup^\flat)$ and $P^\sharp(\sqsubseteq^\sharp, \bot^\sharp, \top^\sharp, \sqcap^\sharp, \sqcup^\sharp)$ be complete lattices with a Galois connection $P^\flat \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} P^\sharp$, $F^\flat \in P^\flat \to P^\flat$ and $B^\flat \in P^\flat \to P^\flat$ be two monotonic functions, and let*

$$L^\flat = \operatorname{gfp} \lambda Z.(Z \sqcap^\flat F^\flat(Z) \sqcap^\flat B^\flat(Z))$$

If $F^\sharp \in P^\sharp \to P^\sharp$ and $B^\sharp \in P^\sharp \to P^\sharp$ are monotonic and satisfy $\alpha \circ F^\flat \circ \gamma \sqsubseteq^\sharp F^\sharp$ and $\alpha \circ B^\flat \circ \gamma \sqsubseteq^\sharp B^\sharp$, then the sequence $(X_n)_{n\in\mathcal{N}}$ defined by $X_0 = \top^\sharp$, $X_{2n+1} = X_{2n} \sqcap^\sharp F^\sharp(X_{2n})$, and $X_{2n+2} = X_{2n+1} \sqcap^\sharp B^\sharp(X_{2n+1}), \forall n \geq 0$ is such that:

$$\forall n \geq 0,\ \alpha(L^\flat) \sqsubseteq^\sharp X_{n+1} \sqsubseteq^\sharp X_n$$

The optimality of this approach has been proved in [1]: it has been shown that $L^\sharp = \operatorname{gfp} \lambda Z.(Z \sqcap^\sharp F^\sharp(Z) \sqcap^\sharp B^\sharp(Z))$ is the greatest lower bound of the set $E^\sharp$ defined inductively as:

- $\top^\sharp \in E^\sharp$
- If Z is in $E^\sharp$ then so are $F^\sharp(Z)$ and $B^\sharp(Z)$.
- If Z_1 and Z_2 are in $E^\sharp$ then so are $Z_1 \sqcap^\sharp Z_2$ and $Z_1 \sqcup^\sharp Z_2$.

Therefore, $L^\sharp$ is the best upper approximation of $\alpha(L^\flat)$ that can be obtained using $F^\sharp$ and $B^\sharp$.

Standard backward-forward combination

The standard backward-forward combination [4] derives from an application of this lemma to the particular case of backward and forward collecting semantics:

$F^\flat$, $B^\flat$, $F^\sharp$ and $B^\flat$ are instantiated as follows:

$$\begin{aligned}
F^\flat &= \lambda Y.\operatorname{lgfp}_1 \lambda X.(Y \sqcap^\flat f^\flat(X)) \\
B^\flat &= \lambda Y.\operatorname{lgfp}_2 \lambda X.(Y \sqcap^\flat b^\flat(X)) \\
F^\sharp &= \lambda Y.\operatorname{lgfp}_1 \lambda X.(Y \sqcap^\sharp f^\sharp(X)) \\
B^\sharp &= \lambda Y.\operatorname{lgfp}_2 \lambda X.(Y \sqcap^\sharp b^\sharp(X))
\end{aligned}$$

where lgfp means either lfp or gfp, and $f^\flat$, $b^\flat \in P^\flat \to P^\flat$, $f^\sharp$, $b^\sharp \in P^\sharp \to P^\sharp$ are monotonic. When $\alpha \circ f^\flat \circ \gamma \sqsubseteq^\sharp f^\sharp$ and $\alpha \circ b^\flat \circ \gamma \sqsubseteq^\sharp b^\sharp$, the conditions of Lemma 1 are satisfied [7].

Now, let $P^\flat = \wp(\Sigma)$, Σ a set of states[3], and $\tau \in \wp(\Sigma \times \Sigma)$ a transition relation. As usual, we define pre, $\widetilde{pre}$, $post$, $\widetilde{post}$ as:

$$\begin{aligned}
post(X) &= \{s' \mid \exists s : \langle s, s'\rangle \in \tau \wedge s \in X\} \\
\widetilde{post}(X) &= \{s' \mid \forall s : \langle s, s'\rangle \in \tau \Rightarrow s \in X\} \\
pre(X) &= \{s \mid \exists s' : \langle s, s'\rangle \in \tau \wedge s' \in X\} \\
\widetilde{pre}(X) &= \{s \mid \forall s' : \langle s, s'\rangle \in \tau \Rightarrow s' \in X\}
\end{aligned}$$

Given $\mathcal{I}, \mathcal{F} \subseteq \Sigma$, sets of initial and final states, $f^\flat = \lambda X.(\mathcal{I} \cup post(X))$ and $b^\flat = \lambda X.(\mathcal{F} \cup pre(X))$ (and $\operatorname{lgfp}_1 = \operatorname{lgfp}_2 = \operatorname{lfp}$), we have [4]:

$$L^\flat = F^\flat(\top^\flat) \sqcap^\flat B^\flat(\top^\flat) = \operatorname{lfp} f^\flat \cap \operatorname{lfp} b^\flat \qquad (1)$$

By computing $\gamma(L^\sharp)$[4], we obtain a good upper approximation of $F^\flat(\Sigma) \cap B^\flat(\Sigma)$ (at least equal to $\gamma(F^\sharp(\top^\sharp) \sqcap^\sharp B^\sharp(\top^\sharp))$).

The ASTRÉE static analyzer

ASTRÉE: A Sound, Automatic, Specializable, Domain-Aware, Parametric, Modular, Efficient and Precise Static Program Analyzer

`www.astree.ens.fr`

- C programs:
  - with
    - pointers (including on functions), structures and arrays
    - floating point computations
    - tests, loops and function calls
    - limited branching (forward `goto`, `break`, `continue`)

- without
  - `union`
  - dynamic memory allocation
  - recursive function calls
  - backward branching
  - conflict side effects
  - C libraries
- Application Domain: safety critical embedded real-time synchronous software for non-linear control of very complex control/command systems.

Concrete Operational Semantics

- International norm of C (ISO/IEC 9899:1999)
- *restricted by* implementation-specific behaviors depending upon the machine and compiler (e.g. representation and size of integers, IEEE 754-1985 norm for floats and doubles)
- *restricted by* user-defined programming guidelines (such as no modular arithmetic for signed integers, even though this might be the hardware choice)
- *restricted by* program specific user requirements (e.g. `assert`)

Abstract Semantics

– Trace-based refinement of the reachable states for the concrete operational semantics
– Volatile environment is specified by a *trusted* configuration file.

Implicit Specification: Absence of Runtime Errors

– No violation of the norm of C (e.g. array index out of bounds)
– No implementation-specific undefined behaviors (e.g. maximum short integer is 32767)
– No violation of the programming guidelines (e.g. static variables cannot be assumed to be initialized to 0)
– No violation of the programmer assertions (must all be statically verified).

Example application

– Primary flight control software of the A340/A380 fly-by-wire system

– C program, automatically generated from a proprietary high-level specification (à la Simulink/SCADE)
– A340 family: 132,000 lines, 75,000 LOCs after preprocessing, 10,000 global variables, over 21,000 after expansion of small arrays
– A380: $\times$ 3

The Class of Considered Periodic Synchronous Programs

```
declare volatile input, state and output variables;
initialize state and output variables;
loop forever
   - read volatile input variables,
   - compute output and state variables,
   - write to volatile output variables;
  wait_for_clock ();
end loop
```

– Requirements: the only interrupts are clock ticks;
– Execution time of loop body less than a clock tick [1].

Reference

[1] C. Ferdinand, R. Heckmann, M. Langenbach, F. Martin, M. Schmidt, H. Theiling, S. Thesing, and R. Wilhelm. Reliable and precise WCET determination for a real-life processor. *ESOP (2001)*, LNCS 2211, 469–485.

Characteristics of the ASTRÉE Analyzer

Static: compile time analysis (≠ run time analysis Rational Purify, Parasoft Insure++)

Program Analyzer: analyzes programs not micromodels of programs (≠ PROMELA in SPIN or Alloy in the Alloy Analyzer)

Automatic: no end-user intervention needed (≠ ESC Java, ESC Java 2)

Sound: covers the whole state space (≠ MAGIC, CBMC) so never omit potential errors (≠ UNO, CMC from coverity.com) or sort most probable ones (≠ Splint)

Characteristics of the ASTRÉE Analyzer (Cont'd)

Multiabstraction: uses many numerical/symbolic abstract domains (≠ symbolic constraints in Bane or the canonical abstraction of TVLA)

Infinitary: all abstractions use infinite abstract domains with widening/narrowing (≠ model checking based analyzers such as VeriSoft, Bandera, Java PathFinder)

Efficient: always terminate (≠ counterexample-driven automatic abstraction refinement BLAST, SLAM)

Characteristics of the ASTRÉE Analyzer (Cont'd)

Specializable: can easily incorporate new abstractions (and reduction with already existing abstract domains) (≠ general-purpose analyzers PolySpace Verifier)

Domain-Aware: knows about control/command (e.g. digital filters) (as opposed to specialization to a mere programming style in C Global Surveyor)

Parametric: the precision/cost can be tailored to user needs by options and directives in the code

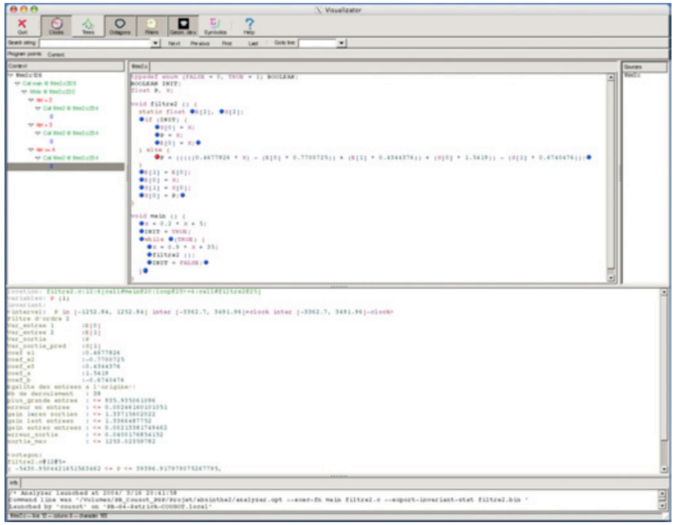
Characteristics of the ASTRÉE Analyzer (Cont'd)

Automatic Parametrization: the generation of parametric directives in the code can be programmed (to be specialized for a specific application domain)

Modular: an analyzer instance is built by selection of O-CAML modules from a collection each implementing an abstract domain

Precise: very few or no false alarm when adapted to an application domain ⟶ it is a VERIFIER!

Example of Analysis Session



Benchmarks (A340 Primary Flight Control Software)

- 132,000 lines, 75,000 LOCs after preprocessing
- Comparative results (commercial software):
  4,200 (false?) alarms,
  3.5 days;
- Our results, November 2003:
  0 alarms,
  40mn on 2.8 GHz PC,
  300 Megabytes
  ⟶ A world première!

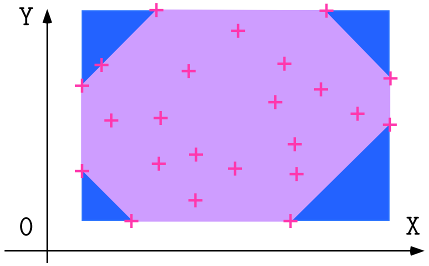
(A380 Primary Flight Control Software)

- 350,000 lines
- 0 alarms (mid-October 2004!),
  7h [2] on 2.8 GHz PC,
  1 Gigabyte
  ⟶ A world grand première!

[2] We are still in a phase where we favour precision rather than computation costs, and this should go down. For example, the A340 analysis went up to 5 h, before being reduced by requiring less precision while still getting no false alarm.

Examples of Abstractions

General-Purpose Abstract Domains: Intervals and Octagons



Intervals:

$$\begin{cases} 1 \leq x \leq 9 \\ 1 \leq y \leq 20 \end{cases}$$

Octagons:

$$\begin{cases} 1 \leq x \leq 9 \\ x + y \leq 77 \\ 1 \leq y \leq 20 \\ x - y \leq 04 \end{cases}$$

Difficulties: many global variables, arrays (smashed or not), IEEE 754 floating-point arithmetic (in program and analyzer)

Floating-Point Computations

– Code Sample:

```
/* float-error.c */
int main () {
  float x, y, z, r;
  x = 1.000000019e+38;
  y = x + 1.0e21;
  z = x - 1.0e21;
  r = y - z;
  printf("%f\n", r);
} % gcc float-error.c
% ./a.out
0.000000
```

$(x+a) - (x-a) \neq 2a$

```
/* double-error.c */
int main () {
double x; float y, z, r;
/* x = ldexp(1.,50)+ldexp(1.,26); */
x = 1125899973951487.0;
y = x + 1;
z = x - 1;
r = y - z;
printf("%f\n", r);
}
% gcc double-error.c
% ./a.out
0.000000
```


Floating-Point Computations

– Code Sample:

```
/* float-error.c */
int main () {
  float x, y, z, r;
  x = 1.000000019e+38;
  y = x + 1.0e21;
  z = x - 1.0e21;
  r = y - z;
  printf("%f\n", r);
} % gcc float-error.c
% ./a.out
0.000000
```

$(x+a) - (x-a) \neq 2a$

```
/* double-error.c */
int main () {
double x; float y, z, r;
/* x = ldexp(1.,50)+ldexp(1.,26); */
x = 1125899973951488.0;
y = x + 1;
z = x - 1;
r = y - z;
printf("%f\n", r);
}
% gcc double-error.c
% ./a.out
134217728.000000
```


Symbolic abstract domain

- Interval analysis: if $x \in [a, b]$, $y \in [c, d]$ & $a, c \geq 0$ then $x - y \in [a - d, b - c]$ so if $x \in [0, 100]$ then $x - x \in [-100, 100]$!!!
- The symbolic abstract domain propagates the symbolic values of variables and performs simplifications;
- Must maintain the maximal possible rounding error for float computations (overestimated with intervals);

```
% cat -n x-x.c
     1  void main () { int X, Y;
     2          __ASTREE_known_fact(((0 <= X) && (X <= 100)));
     3          Y = (X - X);
     4          __ASTREE_log_vars((Y)); }
```

```
astree -exec-fn main -no-relational x-x.c
Call main@x-x.c:1:5-x-x.c:1:9:
<interval: Y in [-100, 100]>
```

```
astree -exec-fn main x-x.c
Call main@x-x.c:1:5-x-x.c:1:9:
<interval: Y in {0}> <symbolic: Y = (X -i X)>
```

Clock Abstract Domain for Counters

– Code Sample:

```
R = 0;
while (1) {
  if (I)
    { R = R+1; }
  else
    { R = 0; }
  T = (R>=n);
  wait_for_clock ();
}
```

- Output T is true iff the volatile input I has been true for the last n clock ticks.
- The clock ticks every s seconds for at most h hours, thus R is bounded.
- To prove that R cannot overflow, we must prove that R cannot exceed the elapsed clock ticks *(impossible using only intervals)*.

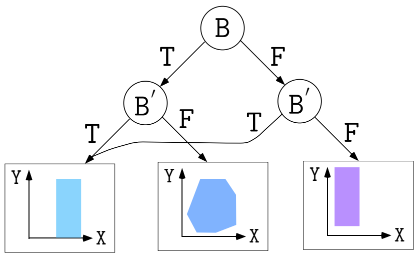
– Solution:

- We add a phantom variable `clock` in the concrete user semantics to track elapsed clock ticks.
- For each variable X, we abstract three intervals: X, X+clock, and X-clock.
- If X+clock or X-clock is bounded, so is X.

Boolean Relations for Boolean Control

– Code Sample:

```
/* boolean.c */
typedef enum {F=0,T=1} BOOL;
BOOL B;
void main () {
  unsigned int X, Y;
  while (1) {
    ...
    B = (X == 0);
    ...
    if (!B) {
      Y = 1 / X;
    }
    ...
  }
}
```



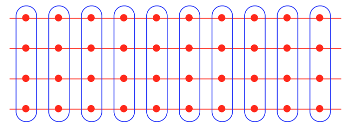
The boolean relation abstract domain is parameterized by the height of the decision tree (an analyzer option) and the abstract domain at the leafs

Control Partitionning for Case Analysis

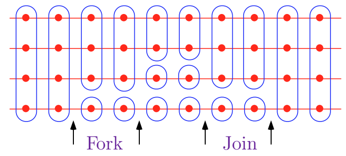
– Code Sample:

```
/* trace_partitionning.c */
void main() {
  float t[5] = {-10.0, -10.0, 0.0, 10.0, 10.0};
  float c[4] = {0.0, 2.0, 2.0, 0.0};
  float d[4] = {-20.0, -20.0, 0.0, 20.0};
  float x, r;
  int i = 0;

  ... found invariant -100 ≤ x ≤ 100 ...

  while ((i < 3) && (x >= t[i+1])) {
    i = i + 1;
  }
  r = (x - t[i]) * c[i] + d[i];
}
```

Control point partitionning:

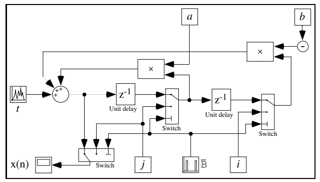


Trace partitionning:

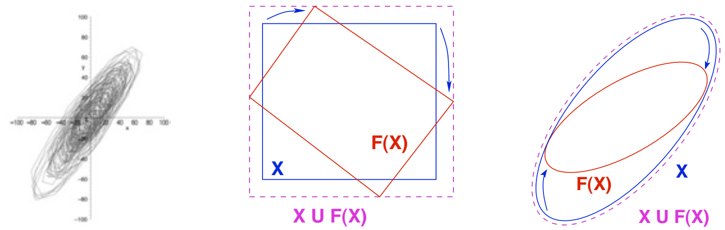


Delaying abstract unions in tests and loops is more precise for non-distributive abstract domains (and much less expensive than disjunctive completion).

2d Order Digital Filter: Ellipsoid Abstract Domain for Filters



- Computes $X_n = \begin{cases} \alpha X_{n-1} + \beta X_{n-2} + Y_n \\ I_n \end{cases}$
- The concrete computation is bounded, which must be proved in the abstract.
- There is no stable interval or octagon.
- The simplest stable surface is an ellipsoid.



execution trace — unstable interval — stable ellipsoid

Filter Example

```
typedef enum {FALSE = 0, TRUE = 1} BOOLEAN;
BOOLEAN INIT; float P, X;
void filter () {
  static float E[2], S[2];
  if (INIT) { S[0] = X; P = X; E[0] = X; }
  else { P = (((((0.5 * X) - (E[0] * 0.7)) + (E[1] * 0.4))
           + (S[0] * 1.5)) - (S[1] * 0.7)); }
  E[1] = E[0]; E[0] = X; S[1] = S[0]; S[0] = P;
  /* S[0], S[1] in [-1327.02698354, 1327.02698354] */
}
void main () { X = 0.2 * X + 5; INIT = TRUE;
  while (1) {
    X = 0.9 * X + 35; /* simulated filter input */
    filter (); INIT = FALSE; }
}
```


Arithmetic-Geometric Progressions

```
% cat retro.c
typedef enum {FALSE=0, TRUE=1} BOOL;
BOOL FIRST;
volatile BOOL SWITCH;
volatile float E;
float P, X, A, B;

void dev( )
{ X=E;
  if (FIRST) { P = X; }
  else
    { P = (P - ((((2.0 * P) - A) - B)
          * 4.491048e-03)); };
  B = A;
  if (SWITCH) {A = P;}
  else {A = X;}
}
```

```
void main()
{ FIRST = TRUE;
  while (TRUE) {
    dev( );
    FIRST = FALSE;
    __ASTREE_wait_for_clock(());
  }}
% cat retro.config
__ASTREE_volatile_input((E [-15.0, 15.0]));
__ASTREE_volatile_input((SWITCH [0,1]));
__ASTREE_max_clock((3600000));
|P| <= (15.  + 5.87747175411e-39
/ 1.19209290217e-07) * (1 +
1.19209290217e-07)^clock -
5.87747175411e-39 / 1.19209290217e-07
<= 23.0393526881
```


(Automatic) Parameterization

- All abstract domains of ASTRÉE are parameterized, e.g.
  - variable packing for octagones and decision trees,
  - partition/merge program points,
  - loop unrollings,
  - thresholds in widenings, ...;
- End-users can either parameterize by hand (analyzer options, directives in the code), or
- choose the automatic parameterization (default options, directives for pattern-matched predefined program schemata).

The main loop invariant for the A340

A textual file over 4.5 Mb with

- 6,900 boolean interval assertions ($x \in [0;1]$)
- 9,600 interval assertions ($x \in [a;b]$)
- 25,400 clock assertions ($x+\mathrm{clk} \in [a;b] \wedge x-\mathrm{clk} \in [a;b]$)
- 19,100 additive octagonal assertions ($a \le x+y \le b$)
- 19,200 subtractive octagonal assertions ($a \le x-y \le b$)
- 100 decision trees
- 60 ellipse invariants, etc ...

involving over 16,000 floating point constants (only 550 appearing in the program text) $\times$ 75,000 LOCs.

Possible origins of imprecision and how to fix it

In case of false alarm, the imprecision can come from:

- Abstract transformers (not best possible) $\longrightarrow$ improve algorithm;
- Automatized parametrization (e.g. variable packing) $\longrightarrow$ improve pattern-matched program schemata;
- Iteration strategy for fixpoints $\longrightarrow$ fix widening [3];
- Inexpressivity i.e. indispensable local inductive invariant are inexpressible in the abstract $\longrightarrow$ add a new abstract domain to the reduced product (e.g. filters).

[3] This can be very hard since at the limit only a precise infinite iteration might be able to compute the proper abstract invariant. In that case, it might be better to design a more refined abstract domain.

Bibliography

[2] P. Cousot. *Méthodes itératives de construction et d'approximation de points fixes d'opérateurs monotones sur un treillis, analyse sémantique de programmes*. Thèse d'État ès sciences mathématiques, Université scientifique et médicale de Grenoble, Grenoble, France, 21 March 1978.

[3] P. Cousot and R. Cousot. Abstract interpretation and application to logic programs. *Journal of Logic Programming*, 13(2–3):103–179, 1992. [4]

[4] www.astree.ens.fr/

[4] The editor of JLP has mistakenly published the unreadable galley proof. For a correct version of this paper, see `http://www.ens.fr/~cousot`.

THE END

My MIT web site is http://www.mit.edu/~cousot/

The course web site is http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/.